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Research Paper

On the Operator Sturm-Liouville Problem with Unbounded Operator Coefficients in Boundary Condition

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Abstract. In this study, we carry out the spectral analysis of operators generated by the Sturm-Liouville equation and boundary conditions, where both contain unbounded operator coefficients and a spectral parameter. Such problems model many processes in classical and quantum mechanics as well as in mechanical engineering. In this paper, we provide a complete description of the domains corresponding to the problem operators in the exit space, a characterization of the spectrum, and the asymptotic behavior of the discrete spectrum. The theoretical results obtained in this study are particularly applicable to areas such as vibration analysis, stability, and mathematical modeling of dynamic systems in mechanical engineering. The study of the spectral characteristics of operators provides an important theoretical basis for determining the natural frequencies and critical loads of thin-walled structures with complex boundary conditions. Furthermore, it provides a general mathematical framework that can be used to model the behavior of engineering systems under thermoelastic, piezoelectric, and magnetoelastic effects. In these respects, the study has the potential to contribute to structural safety, vibration control, and advanced material-based design approaches in mechanical engineering.

Keywords: Operator differential equation, Self-adjoint extension, Spectral parameter dependent boundary condition, Asymptotic of eigenvalues, Sustainable innovation.

1. Introduction

In modern approaches to solid body mechanics, mathematical models developed to investigate the vibration, buckling, and dynamic stability behavior of complex structural elements are often expressed on the basis of operator theory. Within this framework, Sturm–Liouville problems with unbounded operator coefficients are widely used in both theoretical and applied mechanics. Such problems provide an effective mathematical framework for the vibration and stability analyses of plates, as well as for the study of systems operating under elastic/viscoelastic boundary conditions and mixed foundations [1-3].

In this approach, the physical problem is defined on a Hilbert space using a differential operator defined under appropriate boundary conditions. Thus, the dynamic behavior of the system is represented by a Sturm–Liouville-type operator equation. That problem is also modeling dynamic and stability behavior of continuum elements, such as long beam and rod systems, and structures that vary in material properties or cross-sectional geometry.

Spectral analysis is a powerful tool for determining the eigenfrequency distributions of these operators, the asymptotic properties of high-frequency modes, the resonant behavior of the system, and fatigue predictions. Thus, this method allows for detailed elucidation of the spectral distribution and buckling threshold behavior of high-frequency vibrations in functionally graded or nanocomposite structures.

This study addresses the solution of some spectral properties of problems defined in space $L_2(0,1;H)$ for the Sturm–Liouville equations with unbounded operator coefficients. The mathematical model of this problem is defined as follows:

$$ly := -y''(t) + Ay(t) + q(t)y(t) = \lambda y(t) \tag{1}$$

where A is unbounded self-adjoint operator and $q(t)$ is self-adjoint bounded operator in separable Hilbert space H.

Note that coefficient A of the equation might be any differentiation operator, even of fractional order defined via spectral expansion as:

$$A^{\alpha} f = \int_{-\infty}^{+\infty} \lambda^{\alpha} d(E_{\lambda} f, f), \tag{2}$$

where E_λ is the spectral resolution of identity associated with self-adjoint operator A and $0<\alpha<1$.

For example, if A is defined by:

$$Af(x)=\frac{d^2 f}{dx^2} \tag{3}$$

$$D(A)=\left\{f(x)\in L_2(0,\pi),\ Af\in L_2(0,\pi), y(0)=y(1)=0\right\} \tag{4}$$

then

$$A^{\frac{1}{2}}f(x)=\frac{idf}{dx},\ \ D(A)=\left\{f(x)\in W_2^1(0,\pi),\ \ y(0)=y(\pi)\right\}, \tag{5}$$

and

$$A^{\frac{1}{4}}f(x)=\sqrt{i}\frac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}} \quad \text{etc.} \tag{6}$$

Differential-operator settings as in Eq. (1) are excellent models for problems of mathematical-physics and mechanics. For example, let's model the next boundary value problem to set it as eigenvalue problem for Eq. (1) with suitably chosen A^a. For the sake, consider in cylinder $\Omega\times[0,T]$, $\Omega=[0,1]^2$ the problem at $u(t,x,y)$:

$$\frac{\partial u}{\partial t}=-\frac{\partial^2 u}{\partial x^2}+\frac{\partial^4 u}{\partial y^4}+q(x,y)u,\ \ u=u(t,x,y) \tag{7}$$

subject to,

$$u(0,x,y)=0, \tag{8}$$

$$u(t,0,y)=u_{xx}(t,0,y)=0, \tag{9}$$

$$u_x(t,1,y)=u_{tyy}(t,1,y), \tag{10}$$

$$u(t,x,0)=u(t,x,1)=0, \tag{11}$$

$$u_{yy}(t,x,0)=u_{yy}(t,x,1)=0,\ \ x\in[0,1], y\in[0,1], t\in[0,T] \tag{12}$$

By the method of separation of variables, we look for solution in the form $u(t,x,y)=U(t)V(x,y)$, which yields the next eigenvalue problem with eigenvalue parameter λ included also in boundary condition:

$$\frac{\partial^2 V(x,y)}{\partial^2 x^2}+\frac{\partial^4 V(x,y)}{\partial y^4}+q(x,y)V(x,y)=\lambda V(x,y), \tag{13}$$

$$V(0,y)=0, \tag{14}$$

$$V_x(1,y)=\lambda V_y(1,y), \tag{15}$$

$$V(x,0)=V_{yy}(x,0)=V(x,1)=V_{yy}(x,1)=0. \tag{16}$$

If one defines operator A in $L_2(0,1)$ by:

$$Av(y)=v''(y), V(\bullet,y)\equiv v(y) \tag{17}$$

$$D(A)=\left\{v(y)\in\ L_2(0,1),\ \ v'(y)\in AC[0,1],\ v''(y)\in L_2(0,1)\quad v(0)=v(1)=v''(0)=v''(1)=0\right\} \tag{18}$$

and $Q=A$, then operator setting of problem (13)-(16) will be as follows:

$$-u''+Au+qu=\lambda u \tag{19}$$

$$u=0 \tag{20}$$

$$u'(1)=\lambda Qu(1) \tag{21}$$

We will study the spectral properties of operators generated by Eq. (1) but in broader space than $L_2(0,1;H)$, namely in $L_2=L_2(0,1;H)\oplus H^{(N+1)}$.

Such operators arise upon consideration of boundary value or eigenvalue problems with spectral parameter dependent boundary conditions as in above given example. Problems with eigenvalue parameter in equation as well as in boundary conditions arise in different areas such as heat conduction, quantum mechanics, fluid dynamics (see [4] and references therein), for recent applications in biology, mathematical finance, quantum computing see citations in [5]. Refer here to works [6-8] where they studied Sturm-Liouville operator equation with boundary conditions involving spectral parameter λ linearly or rationally and to [9,10] where corresponding spectral analysis was held for fourth order differential operator equations with unbounded operator coefficients.

In current study our aim is to investigate operators in space L_2 generated by Eq. (1) and boundary conditions involving Herglotz-Nevanlinna function in general setting and as well unbounded operator. Detailed analysis of spectral properties of operators generated by Eq. (1) but in the space $L_2(0,1;H)$ is given in the monography by Gorbachuk and Gorbachuk [11]. From [12], it follows that spectrum of any operator L having tensor de-composition $L = A_1 \otimes I + I \otimes A_2$ with self-adjoint operators A_1, A_2 and identity operator I in H is:

$$\sigma(L) = \sigma(A_1) + \sigma(A_2). \tag{22}$$

Hence, if spectrum of coefficient A of Eq. (1) is continuous then spectrum of operators generated by Eq. (1) also will be continuous. But if A is operator with discrete spectrum, then there might be operators generated by Eq. (1) having purely discrete spectrum as well as continuous spectrum depending on conditions at end points of interval (0,1). Our requirement $A^{(-1)} \in \sigma_\infty$ provides discreteness of spectrum of A.

Subject of our study is, as it was indicated above, consideration of operators which will cover boundary value problems for Eq. (1) with boundary condition at end point 1 of type:

$$y'(1) = f(\lambda)Qy(1), \tag{23}$$

where

$$f(\lambda) \equiv a\lambda + b - \sum_{n=1}^{N} \frac{b_n}{\lambda - c_n} \tag{24}$$

as it is proved in [13] is Herglotz-Nevalinna function (one satisfying $f(\bar{z}) = \overline{f(z)}$), if and only if $a \ge 0, b_k > 0, c_1 < c_2 < \ldots < c_N, N \ge 0$. Q is self-adjoint positive-define unbounded operator in H.

Note that problem (1), Eq. (23) but without Q is studied in our recent paper [6]. Existence of unbounded operator Q in boundary condition models boundary conditions for partial differential equations involving differentiation operation at the endpoint of interval. Note that even case with bounded operator in boundary condition in Eq. (23) is not investigated, so our treatment being more general is new from that side too. Change in boundary condition in comparison with [6] gives rise to new operator setting of problem, like defining appropriate space and dot product there and giving necessary and sufficient conditions for existence symmetric minimal operator and its different extensions, nature of spectrum, eigenvectors and eigenvalues. So, we treated all that questions in current study. Refer here to works [13-23] where studied differential equations in scalar case with boundary conditions containing Herglotz-Nevanlinna function of spectral parameter. Recent advances in recovering the stiffness coefficients of Sturm–Liouville operators on complex structures, such as star graphs, have been reported in [24], expanding the practical applications of spectral theory in mathematical physics.

For differential operator equations with unbounded operator coefficients, see [20, 21], where considered regularized traces.

The organization of paper is as follows:

In the introduction section, the statement of problem, motivation and main achievement in comparison with previous works was presented. In section 2, posed operator setting of problem under consideration, as well described domains of definitions of minimal, maximal operators and self-adjoint extensions of minimal operators, treated requirements for discrete spectrum is presented. In section 3, the obtained asymptotic formula for spectrum is provided. Finally, section 4 summarizes the main theoretical results and highlights their potential implications for the analysis and modeling of engineering systems.

2. Operators Generated in Exit Space L_2

Set in exit space $L_2 = L^2(0,1,H) \oplus H^{N+1}$ a scalar product of elements $Y, Z \in L_2$, $Y = (y(t), y_1, \ldots y_{N+1})$, $Z = (z(t), z_1 \ldots z_{N+1})$ by:

$$(Y,Z)_{L_2} = \int_0^1 (y(t),z(t))_H dt + \sum_{n=1}^{N} \frac{(y_n,z_n)_H}{b_n} + \frac{(Q^{-\frac{1}{2}}y_{N+1}, Q^{-\frac{1}{2}}z_{N+1})_H}{a} \tag{25}$$

It easily might be checked that Eq. (25) really defines an inner product. To define a minimal operator corresponding to (1) in L_2 in class of smooth in some sense functions consider $C_0^\infty(0,1;H_\infty)$ which is class of (see Ref. [15]) infinitely many times differentiable functions vanishing in the vicinity of zero and whose values for each $t \in (0,1)$ are from $H_\infty = \bigcap_{j=1}^{\infty} D(A^j)$. Define operators L_0' by:

$$D(L_0') = \{Y \in L_2,\ \text{where}\ y(t) \in C_0^\infty(0,1;H_\infty)\} \tag{26}$$

and

$$L_0'Y = \left\{ly, c_1y_1 - b_1Qy(1), \ldots, c_Ny_N - b_NQy(1), y'(1) - bQy(1) - \sum_{n=1}^{N} y_N\right\}. \tag{27}$$

Set $q = \{q(t), \underbrace{0, \ldots, 0}_{N+1}\}$, where 0 is zero operator in H. Because of boundedness of q in L_2 one might drop it when studying domains of definition of corresponding operators. Differential-operator expression and minimal operator in case $q \equiv 0$, we will denote by l_0 and L_0', respectively.

Further we will need operator $\widetilde{A}$ which is extension of A and maps H to H_{-1}:

$$\widetilde{A}f = g, f \in H, g \in H_{-1} \tag{28}$$

Thus, output g is a generalized element. Recall that (see [15]) H_j is a scale of Hilbert spaces generated by A, which is a closure of $D(A)$ in the norm generated by the inner product:

$$\left(f,g\right)_{H_j}=\left(A^j f,A^j g\right)_H \tag{29}$$

It will be shown that values of $y(t)$ at end point 1 might be not only from H but from broader class of generalized elements H_{-j} with some j.

Denote by L_0 the closure of L_0'. For $Y\in D(L_0)$ obviously holds:

$$y(0)=y'(0)=0. \tag{30}$$

Adjoint L_0^* of operator L_0 call a maximal operator. The next theorem gives the complete description of elements from domain of definition L_0^*.

Theorem 1. The domain $D(L_0^*)$ consists of elements $Y\in L_2$, $Y=\left(y(t),y_1,...,y_n,aQy(1)\right)$, where:

$$y(t)=e^{-\sqrt{\tilde{A}}t}f+e^{-\sqrt{\tilde{A}}(1-t)}g+\frac{1}{2}\int_0^1 e^{-\sqrt{\tilde{A}}(t-s)}A^{-\frac{1}{2}}h(s)ds, \tag{31}$$

$$Qg\in H, g\in H_{\frac{1}{2}}, f\in H_{-\frac{1}{4}},\quad h(s)\in L^2(0,1;H), y_1,\dots,y_N\in H, \tag{32}$$

and

$$L_0^*Y=\left\{\tilde{l}_0 y(t),c_1y_1-b_1Qy(1),...,c_Ny_N-b_NQy(1),y'(1)-bQy(1)-\sum\nolimits_{n=1}^{N}y_n\right\}, \tag{33}$$

$$\tilde{l}_0 y(t)=y''(t)+\tilde{A}y(t). \tag{34}$$

Proof. Due to our recent works [6, 7] when Q is missed in boundary condition one has $g\in H_{\frac{1}{2}}, f\in H_{-\frac{1}{4}}$. In current study, $y_{N+1}=aQy(1)$ that is $Qy(1)\in H$ from which in view of it follows that $Qg\in H$, $g\in H_{\frac{1}{2}}$. Justifications for requirement $f\in H_{\frac{1}{4}}$ are as in [15]. $y(0)$ and $y'(0)$ are distributions from $H_{-\frac{1}{4}}$ and $H_{-\frac{3}{4}}$, respectively. Recall that:

$$y_0=A^{-\frac{1}{4}}y(0),\ y_0{}'=A^{\frac{1}{4}}(y'(0)+\widetilde{A}^{\frac{1}{2}}y(0)) \tag{35}$$

are from H and called regularized values of $y(t)$ and $y'(t)$ at zero.

Lemma 2.1 and corollary 2.1 from our recent work [6] about description of different extension including self-adjoint ones remain true in present work too.

In particular, self-adjoint extensions L_0^s of L_0 are defined by the next boundary conditions only at end point zero:

$$\cos Cy_0'-\sin Cy_0=0 \tag{36}$$

with self-adjoint operator C in space H.

3. About Self-Adjoint Extensions with Discrete Spectrum and Asymptotic Behavior of Spectrum

Theorem 2. If $A^{-1}\in\sigma_\infty$, then spectrum of self-adjoint extensions L_0^s of the minimal operator L_0 is discrete if and only if $\cos C$ and $QA^{-\frac{1}{2}}$ are compact and $y_1,...,y_N\in H_{\frac{1}{2}}$.

Proof. For any normal number λ the resolvent $R_\lambda(L_0^s)$ exists and defined over all space L_2, since non-real λ is from resolvent set of self-adjoint operator L_0^s. That is the resolvent equation for arbitrary $\tilde{h}=(h(t),h_1,...,h_N,h_{N+1})\in L_2$ is:

$$L_0^sY-\lambda Y=\tilde{h} \tag{37}$$

has a unique solution. Write Eq. (37) in open form:

$$-y''(t)+Ay(t)-\lambda y(t)=h(t), \tag{38}$$

$$c_1y_1-b_1Qy(1)-\lambda y_1=h_1, \tag{38-1}$$

$$\dots\dots, \tag{...}$$

$$c_Ny_N-b_NQy(1)-\lambda y_N=h_N, \tag{38-N}$$

$$y'(1)-bQy(1)-\sum_{n=1}^{N}y_N-\lambda y_{N+1}=h_{(N+1)}. \tag{38-N+1}$$

Applying on both sides of Eq. (37) operator $R_\lambda(L_0^s)$, one has:

$$R_\lambda(L_0^s)\tilde{h}=\begin{bmatrix} y(t,\lambda)\\ y_1\\ \vdots\\ y_N\\ aQy(1)\end{bmatrix}\in L_2, \tag{39}$$

where $y(t,\lambda)$ is solution of Eq. (38).

In virtue of $D(L_0^s)\subset D(L_0^*)$, we obtain:

$$y(t,\lambda)=e^{-\sqrt{A-\lambda I}t}f+e^{-\sqrt{A-\lambda I}(1-t)}g+\frac{1}{2}\int_0^1 e^{-\sqrt{A-\lambda I}(t-s)}(A-\lambda I)^{-\frac{1}{2}}h(s)ds, \tag{40}$$

where

$$f\in H_{-\frac{1}{4}},\ g\in H_{\frac{1}{2}},\ Qg\in H. \tag{41}$$

Taking the next fundamental set of solutions of homogenous equation corresponding to Eq. (38) will simplify our calculations:

$$\omega_1(t,\lambda)f,\ \omega_2(t,\lambda)g,\ \ f,g\in H\ \text{ where }\ \omega_1(t,\lambda)=e^{-\sqrt{A-\lambda I}(1-t)}A^{-\frac{1}{2}},\ \ \omega_2(t,\lambda)=\frac{sh\sqrt{A-\lambda I}(1-t)}{\sqrt{A-\lambda I}}A^{\frac{3}{4}}e^{-\sqrt{A}}. \tag{42}$$

It is easy to see that:

$$\omega_1(1,\lambda)=A^{-\frac{1}{2}},\ \ \omega_2(1,\lambda)=0. \tag{43}$$

With that in mind the solutions of nonhomogeneous equations $y(t,\lambda)$ can be represented as:

$$y(t,\lambda)=\omega_1(t,\lambda)f+\omega_2(t,\lambda)g+\int_0^1 G(t,s,\lambda)h(s)ds. \tag{44}$$

Here must be satisfied:

$$QA^{-\frac{1}{2}}f\in H, g\in H,\ G(t,s,\lambda)=\begin{cases} e^{-\sqrt{A-\lambda I}(1-t)}, t\le s\\ e^{-\sqrt{A-\lambda I}(1-s)}\dfrac{sh\sqrt{A-\lambda I}(1-t)}{\sqrt{A-\lambda I}}, s<t\end{cases} \tag{45}$$

obviously,

$$G(1,s,\lambda)=0. \tag{46}$$

Therefore, with that notation:

$$R_\lambda(L_0^s)\tilde{h}=\begin{bmatrix}\omega_1(t,\lambda)f+\omega_2(t,\lambda)g+\int_0^1 G(t,s,\lambda)h(s)ds\\ A^{-\frac{1}{2}}f_1\\ \vdots\\ A^{-\frac{1}{2}}f_N\\ QaA^{-\frac{1}{2}}f\end{bmatrix} \tag{47}$$

or

$$R_\lambda(L_0^s)\tilde{h}=\begin{bmatrix}\omega_1 & \omega_2 & 0 & \cdots & 0\\ 0 & 0 & A^{-\frac{1}{2}} & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & \cdots & \cdots & A^{-\frac{1}{2}}\end{bmatrix}\begin{bmatrix}f\\ g\\ f_1\\ \vdots\\ f_N\end{bmatrix}+\begin{bmatrix}\int_0^1 G(t,s,\lambda)h(s)ds\\ \vdots\\ \vdots\\ \vdots\\ \int_0^1 G'(1,s,\lambda)h(s)ds\end{bmatrix} \tag{48}$$

From that with the same justifications as in [15], we get that resolvent is compact if and only if $A^{-\frac{1}{2}}$, $\cos C\in\sigma_\infty$ and $y_1\ldots y_n\in H_1$.

Note. Statement of theorem is valid also for operators:

$$L_s=L_0^s+q \tag{49}$$

Now in one particular case of boundary condition of Eq. (36) defining self-adjoint extensions L_0^s, we will establish asymptotic formula for spectrum which consists only of eigenvalues.

Setting $C=\pi I/2$ and $Q=A^\alpha$, $0<\alpha<1/2$ in Eqs. (36) and (23), respectively, we get the next problem:

$$-y''(t)+Ay(t)=\lambda y(t), \tag{50}$$

$$y(0)=0, \tag{51}$$

$$y'(1)=\left(a\lambda+b-\sum_{n=1}^{N}\frac{b_n}{\lambda-c_n}\right)Qy(1). \tag{52}$$

Chosen operators C and Q satisfy Theorem2 from previous section. That is spectrum of corresponding operator consists only of eigenvalues of finite multiplicity with possible accumulation point at infinity.

Solution of Eqs. (50) and (51) is:

$$y(t) = sh\sqrt{A-\lambda I}te^{-\sqrt{A}}f, f \in H_{-\frac{1}{4}} \tag{53}$$

Let $\gamma_1 < \gamma_2 \ldots$ be eigenvalues and $\phi_1, \phi_2 \ldots$ are orthonormal eigenvectors of operator A having purely discrete spectrum. Thus, holds the next spectral expansion:

$$A\cdot = \sum_{k=1}^{\infty} \gamma_k (\cdot, \varphi_k)\varphi_k \tag{54}$$

With that spectral expansion in mind, Eq. (52) turns into the next transcendental equations for Fourier coefficients of $y(t)$:

$$\sqrt{\gamma_k - \lambda}ch\sqrt{\gamma_k - \lambda} = sh\sqrt{\gamma_k - \lambda}f(\lambda)\gamma_k^{\alpha}, \; k = \overline{1,\infty} \tag{55}$$

and set:

$$\sqrt{\varphi_k - \lambda} = z, \tag{56}$$

yields the next equations in z:

$$zchz = shz\left(a\left(\gamma_k - z^2\right) + b - \sum_{n=1}^{N} \frac{b_n}{\gamma_k - z^2 - c_n}\right)\gamma_k^{\alpha} \tag{57}$$

having only real and imaginary roots. For real roots denotes x_k one has:

$$x_k \sim \sqrt{\gamma_k} - \frac{1}{2a\gamma_k^{\alpha}} \tag{58}$$

and corresponding to that roots' eigenvalues have the next asymptotic:

$$\lambda_k \sim \frac{1}{2a}\gamma_k^{\frac{1}{2}-\alpha}, k \to \infty. \tag{59}$$

The behavior of imaginary roots is as:

$$y_{k,n} \sim \pi n, n \to \infty \tag{60}$$

Thus, we obtain the next Theorem.

Theorem 3. Eigenvectors of operator L_0^0 form the next two series:

$$\lambda_k \sim \frac{1}{2a}\gamma_k^{\frac{1}{2}-\alpha} \tag{61}$$

$$\lambda_{k,n} = \gamma_k + y_{k,n}^2, \quad y_{k,n} \sim \pi n, n \to \infty. \tag{62}$$

Under assumption $\gamma_k \sim ck^{\beta}$, $k \to \infty$, $\beta > 0$, we have the next statement.

Theorem 3. If $\gamma_k \sim ck^{\beta}$, then eigenvalues of operator L_0^0 have the next asymptotic:

$$\lambda_n \sim \begin{cases} d_1 n^{\frac{2\beta}{2+\beta}}, \beta \geq 2 \\ d_2 n^{\frac{\beta(1-2\alpha)}{2}}, \; \beta < 2 \end{cases} \tag{63}$$

4. Conclusions

In this study, a comprehensive spectral analysis of the operators generated by the Sturm–Liouville equations, which include spectral parameters and unbounded operator coefficients in the differential expression and boundary conditions, is performed. The primary objective of the study is to provide a complete characterization of the domains of the relevant operators in the exit space, to explain the structure of the spectrum, and to reveal the asymptotic behavior of the discrete spectrum. Given operator theoretical formulation of boundary value problem for Sturm-Liouville equation with unbounded operator coefficient and boundary condition depending on Herglotz-Nevanlinna type function of eigenvalue. It is done by introducing direct sum Hilbert space and properly defining there a dot product which allows a linearization of problem as well as associating with problem self-adjoint extensions. Given application to initial-boundary value problems from mathematical physics. Obtained the next main results:

- Found the domains of definition of minimal and maximal operators corresponding to the considered boundary value problem in exit space;
- Given necessary and sufficient conditions for existence self-adjoint extensions of minimal operator in exit space;
- Justified conditions in terms of coefficients of equation and boundary conditions to exist self-adjoint extensions with purely discrete spectrum;
- Established asymptotic formulas for eigenvalues and proved that they fall into two sequences with different behavior at infinity.

In this context, the obtained theoretical results extend the applicability of spectral properties of operator differential equations in applied mechanics as well as in the safety, durability, and performance analyses of engineering structures.

Author Contributions

N.M. Aslanova played a role in methodology, formal analysis, formulation and validation of statements and partially manuscript writing. S.K. Caliskan played role in design, validation, formal analysis, visualization, partial manuscript writing; Kh.M. Aslanov played a role in conceptualization, literature analysis, methodology, formal analysis, visualization and partial manuscript writing; S. Karayel conducted the manuscript editing, validation and interpretation of theoretical results; The manuscript was written thanks to the contribution of all authors. All authors discussed the results, reviewed and approved the final version of the manuscript.

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Conflict of Interest

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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