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Research Paper

Steady-State Oscillations of a Satellite with Passive Attitude Control System in the Magneto-Lorentz Coordinate System

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Abstract. In this paper, an artificial Earth satellite with an electrodynamic attitude control system is considered. Satellite attitude stabilization in the magneto-Lorentz coordinate system is studied. The problem of the analytical construction of the periodic steady-state oscillations of the satellite in the passive control mode is solved. The solution is obtained in the form of a partial sum of trigonometric series. The algorithm for constructing solution is based on the introduction of auxiliary complex variables that make it possible to transform the initial non-autonomous differential equation of the 6th order to an autonomous differential system of the 8th order, for which the procedure for analytically finding the coefficients of a trigonometric series does not present fundamental difficulties, since it is reduced to sequential transformations using recurrent formulas. The constructed steady-state oscillations of the satellite make it possible to correct measurements of on-board instruments and increase the accuracy of information transmitted from the satellite.

Keywords: Satellite, Electrodynamic attitude control, Magneto-Lorentz coordinate system, Stabilization, Steady-state oscillations, Computer simulation.

1. Introduction

For many nonlinear dynamical systems operating under conditions of unsteady force action, the problem of approximate analytical construction of steady-state oscillatory modes is of great practical importance.

In classical rigid body dynamics, stationary oscillatory regimes have been studied for decades. These studies are currently ongoing [1]. The problems of integrability of the equations of motion are studied on the basis of the Euler-Poisson differential equations [2]. Periodic and regular, quasi-periodic motions are of special interest. The stability of stationary motions of the Goryachev-Sretensky gyrostat is investigated in [3]. More and more complex options for applying force to a classic gyrostat are being considered. In particular, the effect of the gyrostat charge on its rotational motion in a magnetic field is being studied [4]. Along with studying the issues of qualitative behavior of solutions (stability, periodicity), quantitative issues related to the explicit construction of solutions are also analyzed. In this case, asymptotic methods based on the introduction of a small parameter are widely used (see [4] and papers cited therein).

For artificial Earth satellites interacting with the Earth's gravitational and magnetic fields and additionally exposed to numerous force factors (aerodynamic, light pressure, etc.), the problem of approximate analytical construction of steady-state oscillatory modes is practically important for the correct processing of measurement information coming from the satellite [5]. At the same time, this problem is quite complex due to the need to take into account numerous disturbances, often close in magnitude and nonlinear in structure [6]. Therefore, the investigation of this problem remains relevant in the conditions of various perturbations and various basic coordinate systems. The latter circumstance is due to the fact that, depending on the purpose of the satellite and the features of the angular stabilization system, the oscillatory modes of operation of the satellite can be studied in various basic coordinate systems [7].

The most well-known and well-adapted for solving a wide range of practically important problems is the orbital coordinate system rotating together with the radius vector of the center of mass of the satellite relative to the center of the Earth [8, 9]. For problems related to exploring the depths of space, the Koenig coordinate system moving translationally [10] is preferable. For satellites interacting with the geomagnetic field and equipped with magnetic control systems (MCS), not only the orbital and Koenig coordinate systems, but also the magnetic and magneto-velocity coordinate systems [11, 12] can be used as the basic ones. In general, MCS have been successfully used for a long time in various problems of satellite attitude dynamics [12-14]. The features, advantages, and disadvantages of MCS are described, for example, in [15, 16]. Lorentz control systems (LCS) [17], similar to MCS in the nature of the control torques generated and based on the use of Lorentz force torques, are also based on the use of the interaction of satellite with the geomagnetic field [18-20].

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The synthesis of MCS and LCS is an electrodynamic method for controlling the angular orientation of satellites, was proposed in [21]. It is based on the simultaneous use of the capabilities of MCS and LCS, which makes it possible to remove the characteristic limitations inherent in both MCS and LCS separately. In the paper [22], this method was optimized to minimize energy costs, and in the papers [10, 23] it was developed and adapted to solve the problems of stabilizing various modes of satellite attitude motion in the orbital and Koenig coordinate systems.

At the same time, the magneto-Lorentz coordinate system, which was first introduced in [24], is a natural coordinate system for satellite attitude stabilization using an electrodynamic control system. This rotating basic coordinate system, associated with the magnetic induction vector $\vec{B}$ of the geomagnetic field, the Lorentz force, and the orbital velocity of the satellite's center of mass, seems convenient for implementing some scanning modes of the Earth's surface. In [25], the issue of the electrodynamic stabilization of the satellite in the magneto-Lorentz coordinate system was studied. A rigorous analytical substantiation of the asymptotic stability of the direct position of the satellite equilibrium in the magneto-Lorentz coordinate system based on the analysis of nonlinear differential equations of motion is given. At the same time, the disturbing effect of the gravitational torque is taken into account and conditions are obtained that allow for a rational choice of variable coefficients of electrodynamic control depending on the parameters of the satellite and its orbit.

However, the issue of the passive implementation of the controlled attitude motion mode of the satellite in the magneto-Lorentz coordinate system, which is natural for the electrodynamic method of controlling, has not yet been considered. This work is aimed at filling this gap. The goal is to construct an analytical solution of nonautonomous differential equations describing small oscillations of a satellite passively stabilized in the magneto-Lorentz coordinate system, and then compare the analytical solution with the results of computer modeling. Since the expected solution can be considered in the future as undisturbed in the study of various applied problems, and therefore is of interest, it seems advisable to exclude external disturbances unrelated to the effect of the geomagnetic field on the satellite. In particular, to exclude perturbations caused by the gravitational torque, it is assumed that the principal central moments of inertia of the satellite are equal ($A = B = C$). From a mathematical point of view, this paper develops a new technique for constructing steady-state oscillatory modes of satellite, which had not previously been used in problems of satellite attitude dynamics.

2. Problem Statement

A satellite is considered, having its center of mass C moving along a circular equatorial Keplerian orbit with a radius of R. It is assumed that the satellite carries an electric charge Q distributed with a density of σ over a certain volume of V and has its intrinsic magnetic moment $\vec{I}$. When the satellite moves relative to the geomagnetic field with magnetic induction $\vec{B}$, the electric charge and the intrinsic magnetic moment cause the Lorentz and magnetic torques $\vec{M}_L$ and $\vec{M}_M$, respectively [8, 17]. Taking into account the notation $\vec{W} = \vec{B} \times \vec{T}$, $\vec{T} = \vec{v} \times \vec{B}$, we introduce, according to [24], the right-oriented system of unit vectors guiding the axes of the magneto-Lorentz coordinate system, hereinafter referred as WBT system (Fig. 1):

$$\vec{w} = \frac{\vec{W}}{|\vec{W}|}, \quad \vec{b} = \frac{\vec{B}}{|\vec{B}|}, \quad \vec{t} = \frac{\vec{T}}{|\vec{T}|}. \tag{1}$$

It's worth noting that in all cases of the satellite's orbital motion, with the exception of geostationary orbits or close to them, the vector $\vec{T}$ is nonzero (even for orbits with an inclination of $i > 79$ degrees, when the normal component of the geomagnetic field moment can become zero). So, the vectors $\vec{w}, \vec{b}$ and $\vec{t}$ are defined correctly.

The orbital coordinate system $C\xi\eta\zeta$ is introduced so that the axis $C\zeta$ is directed along the local vertical, the axis $C\xi$ along the positive transversal, the axis $C\eta$ along the normal to the plane of the satellite's orbit. The matrix of direction cosines of WBT axes in $C\xi\eta\zeta$ axes is $\mathcal{B} = (b_{ij})^\top$, $(i, j = 1,2,3)$.

The position of satellite principal central axes of inertia $Cxyz$ regarding to WBT coordinate system is represented in "airplane" angles φ, θ, ψ (Fig. 2).

Matrix $\mathcal{N} = (n_{ij})^\top$, $(i, j = 1,2,3)$ of direction cosines of $Cxyz$ axes in WBT takes the following form when position angles are small:

$$\mathcal{N} \approx \begin{pmatrix} 1 & -\psi & \theta \\ \psi & 1 & -\varphi \\ -\theta & \varphi & 1 \end{pmatrix}. \tag{2}$$

The direction cosines matrix of $C\xi\eta\zeta$ axes in $Cxyz$ axes is the following product of matrices: $\mathcal{N}^\top \mathcal{B}^\top$. The columns of this matrix represent coordinates of unit vectors $\vec{\xi}, \vec{\eta}, \vec{\zeta}$ in $Cxyz$ axes.

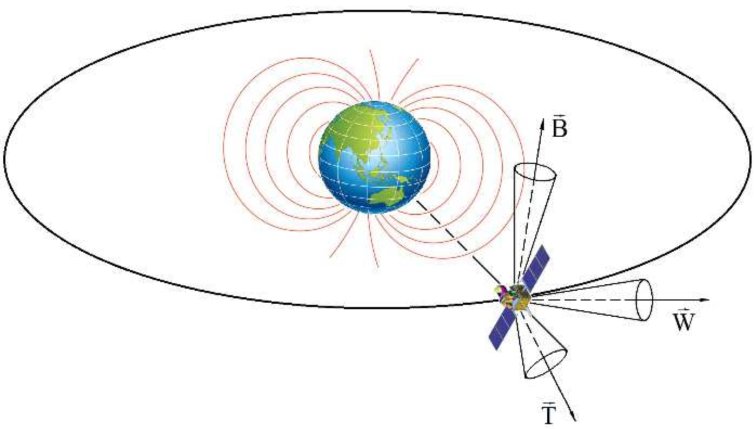

Fig. 1. Satellite in the WBT coordinate system.

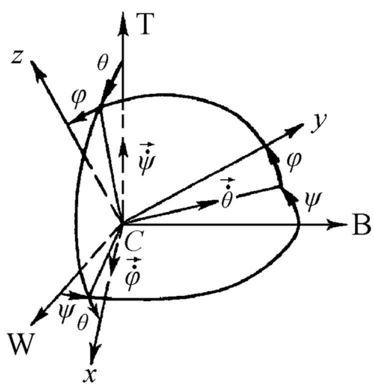

Fig. 2. "Airplane" angles.

3. Kinematic Equations

Let $\vec{\omega}$ be the absolute angular velocity of the satellite, $\vec{\omega'}$ be the angular velocity of the satellite relative to the WBT coordinate system, $\vec{\omega_0'}$ be the angular velocity of the satellite relative to the orbital coordinate system, $\vec{\omega}_b$ be the angular velocity of the WBT coordinate system relative to the Koenig coordinate system, and $\vec{\omega}_{b_0}$ be the angular velocity of the WBT coordinate system relative to the orbital coordinate system. Then $\vec{\omega} = \vec{\omega}_0 + \vec{\omega_0'}$, $\vec{\omega_0'} = \vec{\omega}_{b_0} + \vec{\omega'}$, $\vec{\omega} = \vec{\omega}_0 + \vec{\omega}_{b_0} + \vec{\omega'}$. In addition, according to [25], there are equalities:

$$\vec{\omega}_b = \vec{\omega}_{b_0} + \vec{\omega}_0 = \begin{pmatrix} b_{31}\dot{b}_{21} + b_{32}\dot{b}_{22} + b_{33}\dot{b}_{23} + b_{12}\omega_0 \\ b_{11}\dot{b}_{31} + b_{12}\dot{b}_{32} + b_{13}\dot{b}_{33} + b_{22}\omega_0 \\ b_{21}\dot{b}_{11} + b_{22}\dot{b}_{12} + b_{23}\dot{b}_{13} + b_{32}\omega_0 \end{pmatrix}. \tag{3}$$

According the transition theorem:

$$\left.\frac{d\vec{\omega}}{dt}\right|_{xyz} = \left.\frac{d\vec{\omega}}{dt}\right|_{WBT} - \vec{\omega'} \times \vec{\omega}\,. \tag{4}$$

At the same time:

$$\left.\frac{d\vec{\omega}_b}{dt}\right|_{xyz} = \left.\frac{d\vec{\omega}_b}{dt}\right|_{WBT} - \vec{\omega'} \times \vec{\omega_b}, \quad \left.\frac{d\vec{\omega}}{dt}\right|_{xyz} = \left.\frac{d\vec{\omega}_b}{dt}\right|_{xyz} + \left.\frac{d\vec{\omega'}}{dt}\right|_{xyz}. \tag{5}$$

By definition, in WBT axes:

$$\left.\frac{d\vec{\omega}_b}{dt}\right|_{WBT} = \begin{pmatrix} \dot{\omega}_{b_1} \\ \dot{\omega}_{b_2} \\ \dot{\omega}_{b_3} \end{pmatrix}. \tag{6}$$

Therefore, taking into account Eq. (2), in the $Cxyz$ axes we get, in the first approximation, the following equalities:

$$\begin{aligned} \left.\frac{\mathrm{d}\vec{\omega}_\mathrm{b}}{\mathrm{dt}}\right|_{\mathrm{xyz}} &= \mathcal{N}^{\mathcal{T}} \begin{pmatrix} \dot{\omega}_{b_1} \\ \dot{\omega}_{b_2} \\ \dot{\omega}_{b_3} \end{pmatrix} - \begin{pmatrix} \omega_y'\omega_{bz} - \omega_z'\omega_{by} \\ \omega_z'\omega_{bx} - \omega_x'\omega_{bz} \\ \omega_x'\omega_{by} - \omega_y'\omega_{bx} \end{pmatrix} = \begin{pmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \varphi \\ \theta & -\varphi & 1 \end{pmatrix} \begin{pmatrix} \dot{\omega}_{b_1} \\ \dot{\omega}_{b_2} \\ \dot{\omega}_{b_3} \end{pmatrix} - \begin{pmatrix} \omega_y'\omega_{bz} - \omega_z'\omega_{by} \\ \omega_z'\omega_{bx} - \omega_x'\omega_{bz} \\ \omega_x'\omega_{by} - \omega_y'\omega_{bx} \end{pmatrix} \\ &= \begin{pmatrix} \dot{\omega}_{b_1} + \psi\dot{\omega}_{b_2} - \theta\dot{\omega}_{b_3} - \omega_y'\omega_{bz} + \omega_z'\omega_{by} \\ -\psi\dot{\omega}_{b_1} + \dot{\omega}_{b_2} + \varphi\dot{\omega}_{b_3} - \omega_z'\omega_{bx} + \omega_x'\omega_{bz} \\ \theta\dot{\omega}_{b_1} - \varphi\dot{\omega}_{b_2} + \dot{\omega}_{b_3} - \omega_x'\omega_{by} + \omega_y'\omega_{bx} \end{pmatrix}. \end{aligned} \tag{7}$$

Note that the angular velocity $\vec{\omega'}$ in "airplane" angles has the following form [21]:

$$\vec{\omega'} = \begin{pmatrix} \dot{\varphi} - \dot{\psi}\,sin\,\theta \\ \dot{\theta}\,cos\,\varphi + \dot{\psi}\,sin\,\varphi\,cos\,\theta \\ \dot{\psi}\,cos\,\varphi\,cos\,\theta - \dot{\theta}\,sin\,\varphi \end{pmatrix}. \tag{8}$$

Let us revisit the definition of the local derivative to derive approximate value of the vector $\frac{d\vec{\omega'}}{dt}|_{xyz}$ with up to the second-order precision, represented in terms of angles and angular velocities:

$$\left.\frac{d\vec{\omega'}}{dt}\right|_{xyz} \approx \begin{pmatrix} \ddot{\varphi} - \ddot{\psi}\,sin\,\theta \\ \ddot{\theta}\,cos\,\varphi + \ddot{\psi}\,sin\,\varphi\,cos\,\theta \\ \ddot{\psi}\,cos\,\theta\,cos\,\varphi - \ddot{\theta}\,sin\,\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\,sin\,\theta \\ 0 & cos\,\varphi & sin\,\varphi\,cos\,\theta \\ 0 & -\,sin\,\varphi & cos\,\theta\,cos\,\varphi \end{pmatrix} \begin{pmatrix} \ddot{\varphi} \\ \ddot{\theta} \\ \ddot{\psi} \end{pmatrix}. \tag{9}$$

The derived equations will be used to derive differential equations of small-amplitude oscillations of the satellite in WBT axes.

4. Dynamic Equations

Let M_x, M_y, M_z be the projections on $Cxyz$ axes of the principal torque vector $\vec{M} = \vec{M}_L + \vec{M}_M$ of electromagnetic forces acting on the satellite. Since dynamic Euler equations:

$$\left.\frac{d\vec{\omega}}{dt}\right|_{xyz} = \begin{pmatrix} -[(C-B)\omega_y\omega_z - M_x]/A \\ -[(A-C)\omega_x\omega_z - M_y]/B \\ -[(B-A)\omega_x\omega_y - M_z]/C \end{pmatrix}, \tag{10}$$

in the terms of "airplane" angles, can be reduced to the form $\frac{d\vec{\omega'}}{dt}|_{xyz} = \vec{F}(t,\varphi,\theta,\psi,\dot{\varphi},\dot{\theta},\dot{\psi})$, then based on Eq. (9), the required system of differential equations will take the form:

$$\begin{pmatrix} \ddot{\varphi} \\ \ddot{\theta} \\ \ddot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\varphi & \sin\varphi\cos\theta \\ 0 & -\sin\varphi & \cos\theta\cos\varphi \end{pmatrix}^{-1} \vec{F}(t,\varphi,\theta,\psi,\dot{\varphi},\dot{\theta},\dot{\psi}) = \frac{1}{\cos\theta} \begin{pmatrix} \cos\theta & \sin\theta\sin\varphi & \cos\varphi\sin\theta \\ 0 & \cos\theta\cos\varphi & -\sin\varphi\cos\theta \\ 0 & \sin\varphi & \cos\theta \end{pmatrix} \vec{F}(t,\varphi,\theta,\psi,\dot{\varphi},\dot{\theta},\dot{\psi}). \tag{11}$$

Taking into account only linear terms of this differential equation, we arrive at the following:

$$\begin{pmatrix} \ddot{\varphi} \\ \ddot{\theta} \\ \ddot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \theta \\ 0 & 1 & -\varphi \\ 0 & \varphi & 1 \end{pmatrix} \vec{F}(t,\varphi,\theta,\psi,\dot{\varphi},\dot{\theta},\dot{\psi}). \tag{12}$$

The expression of $\vec{F}(t,\varphi,\theta,\psi,\dot{\varphi},\dot{\theta},\dot{\psi})$ is derived from dynamic Euler equations for the case $A=B=C$ considered below in the form:

$$\left.\frac{d\vec{\omega}}{dt}\right|_{xyz} = \frac{1}{A}\begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix}. \tag{13}$$

Given Eqs. (4) and (5), we get:

$$\left.\frac{d\vec{\omega'}}{dt}\right|_{xyz} + \begin{pmatrix} \dot{\omega}_{b_1} + \psi\dot{\omega}_{b_2} - \theta\dot{\omega}_{b_3} - \omega'_y\omega_{bz} + \omega'_z\omega_{by} \\ -\psi\dot{\omega}_{b_1} + \dot{\omega}_{b_2} + \varphi\dot{\omega}_{b_3} - \omega'_z\omega_{bx} + \omega'_x\omega_{bz} \\ \theta\dot{\omega}_{b_1} - \varphi\dot{\omega}_{b_2} + \dot{\omega}_{b_3} - \omega'_x\omega_{by} + \omega'_y\omega_{bx} \end{pmatrix} = \frac{1}{A}\begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix}. \tag{14}$$

Then,

$$\vec{F}(t,\varphi,\theta,\psi,\dot{\varphi},\dot{\theta},\dot{\psi}) = \frac{1}{A}\begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} - \begin{pmatrix} \dot{\omega}_{b_1} + \psi\dot{\omega}_{b_2} - \theta\dot{\omega}_{b_3} - \omega'_y\omega_{bz} + \omega'_z\omega_{by} \\ -\psi\dot{\omega}_{b_1} + \dot{\omega}_{b_2} + \varphi\dot{\omega}_{b_3} - \omega'_z\omega_{bx} + \omega'_x\omega_{bz} \\ \theta\dot{\omega}_{b_1} - \varphi\dot{\omega}_{b_2} + \dot{\omega}_{b_3} - \omega'_x\omega_{by} + \omega'_y\omega_{bx} \end{pmatrix}. \tag{15}$$

Note that in the $Cxyz$ axes:

$$\vec{\omega_b} = \mathcal{N}^{\mathcal{T}}\begin{pmatrix} \omega_{b_1} \\ \omega_{b_2} \\ \omega_{b_1} \end{pmatrix} = \begin{pmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \varphi \\ \theta & -\varphi & 1 \end{pmatrix}\begin{pmatrix} \omega_{b_1} \\ \omega_{b_2} \\ \omega_{b_1} \end{pmatrix} = \begin{pmatrix} \omega_{b_1} + \psi\omega_{b_2} - \theta\omega_{b_3} \\ -\psi\omega_{b_1} + \omega_{b_2} + \varphi\omega_{b_3} \\ \theta\omega_{b_1} - \varphi\omega_{b_2} + \omega_{b_3} \end{pmatrix}. \tag{16}$$

In addition, since $\vec{\omega} = \vec{\omega}_b + \vec{\omega'}$, then:

$$\vec{\omega} = \begin{pmatrix} \dot{\varphi} + \omega_{b_1} + \psi\omega_{b_2} - \theta\omega_{b_3} \\ \dot{\theta} - \psi\omega_{b_1} + \omega_{b_2} + \varphi\omega_{b_3} \\ \dot{\psi} + \theta\omega_{b_1} - \varphi\omega_{b_2} + \omega_{b_3} \end{pmatrix}. \tag{17}$$

Let's denote $g_i = \omega_{b_i}$. Then, Eqs. (12) and (15) for the case $A=B=C$ take the form:

$$\begin{aligned} \ddot{\varphi} &= \frac{1}{A}M_x + \frac{\theta}{A}M_z - \dot{g}_1 - \psi\dot{g}_2 + \dot{\theta}g_3 - \dot{\psi}g_2, \\ \ddot{\theta} &= \frac{1}{A}M_y - \frac{\varphi}{A}M_z + \psi\dot{g}_1 - \dot{g}_2 + \dot{\psi}g_1 - \dot{\varphi}g_3, \\ \ddot{\psi} &= \frac{1}{A}M_z + \frac{\varphi}{A}M_y - \theta\dot{g}_1 - \dot{g}_3 + \dot{\varphi}g_2 - \dot{\theta}g_1. \end{aligned} \tag{18}$$

5. Torque of Electrodynamic Forces

The idea of electrodynamic control is based on the assumption that the controlled vector $\vec{P}$ of satellite electric charge dipole moment and vector $\vec{I}$ of satellite magnetic moment contain restoring and dissipative components. So the $\vec{P}$ is generated as a sum $\vec{P} = \vec{P}_R + \vec{P}_D$, where $\vec{P}_R = \frac{k_{L_0}}{|\vec{T}|}\vec{k}$, and $\vec{P}_D = \frac{h_{L_0}}{|\vec{T}|^2}\vec{\omega'} \times \vec{T}$ are the restoring and dissipative components of $\vec{P}$ respectively, k_{L_0} and h_{L_0} are scalar multipliers selected at the stage of control synthesis, and $\vec{k}$ is the unit vector of the Cz axis. Then the torque of Lorentz forces takes the form:

$$\vec{M}_L = \vec{M}_{L_R} + \vec{M}_{L_D} = \frac{k_{L_0}}{|\vec{T}|}\vec{k} \times \vec{T} + \frac{h_{L_0}}{|\vec{T}|^2}(\vec{\omega'} \times \vec{T}) \times \vec{T}. \tag{19}$$

In projections on the x,y,z axes, we have:

$$\vec{M}_{L_R} = \frac{k_{L_0}}{|\vec{T}|}\vec{k} \times \vec{T} = k_{L_0}\vec{k} \times \vec{t} = k_{L_0}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} n_{31} \\ n_{32} \\ n_{33} \end{pmatrix} = k_{L_0}\begin{pmatrix} -n_{32} \\ n_{31} \\ 0 \end{pmatrix}, \tag{20}$$

$$\vec{M}_{L_D} = \frac{h_{L_0}}{|\vec{T}|^2}(\vec{\omega'} \times \vec{T}) \times \vec{T} = h_{L_0}\vec{t} \times (\vec{t} \times \vec{\omega'}) = h_{L_0}[\vec{t}(\vec{t}\cdot\vec{\omega'}) - \vec{\omega'}] = h_{L_0}\left[(\omega'_x n_{31} + \omega'_y n_{32} + \omega'_z n_{33})\begin{pmatrix} n_{31} \\ n_{32} \\ n_{33} \end{pmatrix} - \begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix}\right], \tag{21}$$

$$\vec{M}_L = k_{L_0}\begin{pmatrix} -n_{32} \\ n_{31} \\ 0 \end{pmatrix} + h_{L_0}\left[(\omega'_x n_{31} + \omega'_y n_{32} + \omega'_z n_{33})\begin{pmatrix} n_{31} \\ n_{32} \\ n_{33} \end{pmatrix} - \begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix}\right]. \tag{22}$$

Using the expressions (1) and (5) and discarding the values of the second order of smallness, we obtain:

$$\vec{M}_L = k_{L_0}\begin{pmatrix} -\varphi \\ -\theta \\ 0 \end{pmatrix} + h_{L_0}\left[\begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} - \begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}\right] = -\begin{pmatrix} k_{L0}\varphi + h_{L0}\dot{\varphi} \\ k_{L0}\theta + h_{L0}\dot{\theta} \\ 0 \end{pmatrix}. \tag{23}$$

The control torque $\vec{M}_M$ is constructed similarly. The controlled vector $\vec{I}$ is generated as $\vec{I} = \overrightarrow{I_R} + \overrightarrow{I_D}$, where $\overrightarrow{I_R} = \frac{k_{M_0}}{|\vec{B}|}\vec{\jmath}$, and $\overrightarrow{I_D} = \frac{h_{M_0}}{|\vec{B}|^2}\overrightarrow{\omega'} \times \vec{B}$ are the restoring and dissipative components of the vector $\vec{I}$ respectively, k_{M_0} and h_{M_0} are the scalar multipliers, and $\vec{\jmath}$ is the unit vector of the Cy axis. Then the torque of magnetic forces in linear approximation takes the form:

$$\vec{M}_M = -\begin{pmatrix} k_{M_0}\varphi + h_{M_0}\dot{\varphi} \\ 0 \\ k_{M_0}\psi + h_{M_0}\dot{\psi} \end{pmatrix}. \tag{24}$$

Based on Eqs. (11) and (12), the principal moment $\vec{M}$ of electrodynamic forces in linear approximation has the form:

$$\vec{M} = -\begin{pmatrix} (k_{L_0} + k_{M_0})\varphi + (h_{L_0} + h_{M_0})\dot{\varphi} \\ k_{L_0}\theta + h_{L_0}\dot{\theta} \\ k_{M_0}\psi + h_{M_0}\dot{\psi} \end{pmatrix}. \tag{25}$$

When substituting Eq. (13) to the system (10):

$$\begin{pmatrix} \ddot{\varphi} \\ \ddot{\theta} \\ \ddot{\psi} \end{pmatrix} + \begin{pmatrix} H_M + H_L & -g_3 & g_2 \\ g_3 & H_L & -g_1 \\ -g_2 & g_1 & H_M \end{pmatrix}\begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} + \begin{pmatrix} K_M + K_L & 0 & \dot{g}_2 \\ 0 & K_L & -\dot{g}_1 \\ 0 & \dot{g}_1 & K_M \end{pmatrix}\begin{pmatrix} \varphi \\ \theta \\ \psi \end{pmatrix} - K_M\begin{pmatrix} 0 \\ \varphi\psi \\ 0 \end{pmatrix} = -\begin{pmatrix} \dot{g}_1 \\ \dot{g}_2 \\ \dot{g}_3 \end{pmatrix}, \tag{26}$$

where $H_M = h_{M_0}/A, H_L = h_{L_0}/A, K_M = k_{M_0}/A, K_L = k_{L_0}/A$. Numerical research has shown that the nonlinear term $K_M\varphi\psi$ has the same order as the linear terms, therefore its accounting is necessary. This is the only one nonlinear term that should be taken into account. This is caused by the fact that the dimensionless coefficient $\mathcal{K}_{\mathcal{M}} = K_M/\Delta\omega^2$ is of the order of 10, and the product of amplitudes of oscillations of φ and ψ angles is of the same order of smallness as the amplitude of oscillations of the θ angle.

As we will see later, this circumstance will not interfere with the construction of an approximate analytical solution. The coefficients in the equations of the system (14) depend significantly on the model of the geomagnetic field and the parameters of the satellite's orbit. For the purpose of this initial investigation, we will assume that the geomagnetic field is represented by an "inclined magnetic dipole", and the satellite's orbit is circular and equatorial.

6. Calculation of g_i Functions for the Case of a Circular Equatorial Orbit

Let's find the transition matrix $\mathcal{B} = (b_{ij})^\top$ from $C\xi\eta\zeta$ to the WBT coordinate system. To achieve this, we first express the components of the vectors $\vec{W}$ and $\vec{T}$ in terms of the components of the vector $\vec{B}$. For the case of a circular orbit, $\overrightarrow{v_c} = v_c\,\vec{\xi}$. Therefore:

$$\begin{gathered} T_\xi = 0, \quad T_\eta = -v_c B_\zeta, \quad T_\zeta = v_c B_\eta, \\ W_\xi = v_c(B_\eta^2 + B_\zeta^2), \quad W_\eta = -v_c B_\eta B_\xi, \quad W_\zeta = -v_c B_\xi B_\zeta. \end{gathered} \tag{27}$$

Thus, the desired transition matrix takes the following form:

$$\mathcal{B} = \begin{pmatrix} \dfrac{v_c(B_\eta^2 + B_\zeta^2)}{|\vec{B}||\vec{T}|} & \dfrac{B_\xi}{|\vec{B}|} & 0 \\ -\dfrac{v_c B_\eta B_\xi}{|\vec{B}||\vec{T}|} & \dfrac{B_\eta}{|\vec{B}|} & -\dfrac{v_c B_\zeta}{|\vec{T}|} \\ -\dfrac{v_c B_\xi B_\zeta}{|\vec{B}||\vec{T}|} & \dfrac{B_\zeta}{|\vec{B}|} & \dfrac{v_c B_\eta}{|\vec{T}|} \end{pmatrix}. \tag{28}$$

In projections on the axes of the orbital coordinate system, the magnetic induction of the geomagnetic field has the form [8]:

$$\begin{gathered} B_\xi = (R_E/R)^3(g_1^1 \sin u_0 - h_1^1 \cos u_0), \\ B_\eta = -(R_E/R)^3 g_1^0, \\ B_\zeta = 2(R_E/R)^3(g_1^1 \cos u_0 + h_1^1 \sin u_0). \end{gathered} \tag{29}$$

Note that it is possible to simplify Eq. (29) by introducing an auxiliary angle $\gamma = u_0 - \arctan(h_1^1/g_1^1)$ and using the equalities:

$$g_1^1 \sin u_0 - h_1^1 \cos u_0 = \beta \sin\gamma, \qquad g_1^1 \cos u_0 + h_1^1 \sin u_0 = \beta\cos\gamma, \tag{30}$$

where $g_1^1 = -1671.8\ nT,\ h_1^1 = 5080.0\ nT,\ \beta = \sqrt{(g_1^1)^2 + (h_1^1)^2} = 5348.0\ nT$. Taking into account the notation $\alpha = -g_1^0 = 29556.8\ nT,\ \delta = (R_E/R)^3$, we can convert Eq. (15) to the form:

$$B_\xi = \delta\beta\, sin\,\gamma\,, \quad B_\eta = \delta\alpha, \quad B_\zeta = 2\delta\beta\, cos\,\gamma. \tag{31}$$

Further to calculate g_1, g_2 and g_3, we introduce the notation:

$$\begin{gathered} \Pi = \sqrt{\alpha^2 + \beta^2} = 30036.7\ nT, \qquad \Delta_1 = \frac{3\beta^2}{\alpha^2 + \beta^2} = 0.095104\ nT, \qquad \Delta_2 = \frac{4\beta^2}{\alpha^2} = 0.13096\ nT, \\ \Theta_1(\gamma) = \frac{1}{\sqrt{1 + \Delta_1\cos\gamma^2}}, \qquad \Theta_2(\gamma) = \frac{1}{\sqrt{1 + \Delta_2\cos\gamma^2}}. \end{gathered} \tag{32}$$

Then g_1, g_2, and g_3 can be represented (see formulas (A.9), (A.10) and (A.11) in Appendix) as:

$$g_1 = -(3\omega_0 - 2\omega_E)\frac{\beta}{\Pi} sin(\gamma)\ \Theta_1(\gamma)\ \Theta_2(\gamma), \tag{33}$$

$$g_2 = -\frac{2\beta^2(\omega_0-\omega_E)}{\Pi\alpha} sin^2(\gamma)\,\Theta_1(\gamma)\Theta_2^2(\gamma) + \frac{\alpha\omega_0}{\Pi}\Theta_1(\gamma), \tag{34}$$

$$g_3 = -\frac{\beta(4\beta^2+\alpha^2)(\omega_0-\omega_E)}{\alpha\Pi^2} cos(\gamma)\ \Theta_1^2(\gamma)\Theta_2(\gamma) - \frac{2\beta\omega_0}{\alpha} cos\,\gamma\ \Theta_2(\gamma). \tag{35}$$

Periodic functions of γ included in the expressions (33), (34), (35) can be approximated by the following partial sums of Fourier series with an accuracy of 10^{-6}:

$$\begin{gathered}
\sin\gamma\ \Theta_1(\gamma)\ \Theta_2(\gamma) = 97325\cdot10^{-5}\sin\gamma - 26025\cdot10^{-6}\sin3\gamma + 70382\cdot10^{-8}\sin5\gamma - 19242\cdot10^{-9}\sin7\gamma,\\
\sin^2(\gamma)\ \Theta_1(\gamma)\ \Theta_2^2(\gamma) = 47922\cdot10^{-5} - 49927\cdot10^{-5}\cos2\gamma + 20751\cdot10^{-6}\cos4\gamma - 73048\cdot10^{-8}\cos6\gamma + 24216\cdot10^{-9}\cos8\gamma,\\
\Theta_1(\gamma) = 97742\cdot10^{-5} - 22197\cdot10^{-6}\cos2\gamma + 37805\cdot10^{-8}\cos4\gamma - 71542\cdot10^{-10}\cos6\gamma,\\
\cos\gamma\ \Theta_1^2(\gamma)\ \Theta_2(\gamma) = 89177\cdot10^{-5}\cos\gamma - 34171\cdot10^{-6}\cos3\gamma + 10962\cdot10^{-7}\cos5\gamma - 33093\cdot10^{-9}\cos7\gamma,\\
\cos\gamma\ \Theta_2(\gamma) = 95456\cdot10^{-5}\cos\gamma - 14567\cdot10^{-6}\cos3\gamma + \cos\gamma\ \Theta_2(\gamma)\\
= 95456\cdot10^{-5}\cos\gamma - 14567\cdot10^{-6}\cos3\gamma + 33516\cdot10^{-8}\cos5\gamma - 85792\cdot10^{-10}\cos7\gamma.
\end{gathered} \tag{36}$$

Numerical investigation confirms that the derivatives of functions g_1, g_2, and g_3 with respect of γ are equal to the derivatives of the partial sums found with the specified accuracy.

7. Reducing a System of Equations to a Dimensionless Form

Let's replace in the system (14) derivatives by time t to derivatives by the dimensionless variable γ. Considering that $\dot{\gamma} = \omega_0 - \omega_E$ and introducing the notation $\omega_0 - \omega_E = \Delta\omega$, $G_i = g_i/\Delta\omega$, $\frac{d(\ )}{d\gamma} = (\ \ )'$, $\mathcal{H}_\mathcal{M} = H_M/\Delta\omega = \frac{h_{M_0}}{\Delta\omega A}$, $\mathcal{K}_\mathcal{M} = K_M/\Delta\omega^2 = \frac{k_{M_0}}{\Delta\omega^2 A}$, $\mathcal{H}_\mathcal{L} = H_L/\Delta\omega = \frac{h_{L_0}}{\Delta\omega\ A}$, $\mathcal{K}_\mathcal{L} = K_L/\Delta\omega^2 = \frac{k_{L_0}}{\Delta\omega^2 A}$, we obtain:

$$\begin{pmatrix}\varphi''\\ \theta''\\ \psi''\end{pmatrix} + \begin{pmatrix}\mathcal{H}_\mathcal{M}+\mathcal{H}_\mathcal{L} & -G_3 & G_2\\ G_3 & \mathcal{H}_\mathcal{L} & -G_1\\ -G_2 & G_1 & \mathcal{H}_\mathcal{M}\end{pmatrix}\begin{pmatrix}\varphi'\\ \theta'\\ \psi'\end{pmatrix} + \begin{pmatrix}\mathcal{K}_\mathcal{M}+\mathcal{K}_\mathcal{L} & 0 & G_2'\\ 0 & \mathcal{K}_\mathcal{L} & -G_1'\\ 0 & G_1' & \mathcal{K}_\mathcal{M}\end{pmatrix}\begin{pmatrix}\varphi\\ \theta\\ \psi\end{pmatrix} - K_M\begin{pmatrix}0\\ \varphi\psi\\ 0\end{pmatrix} + \begin{pmatrix}G_1'\\ G_2'\\ G_3'\end{pmatrix} = 0. \tag{37}$$

The problem of passive implementation of the oriented motion of the satellite in the WBT coordinate system will be carried out if values $k_{L_0}, h_{L_0}, k_{M_0}, h_{M_0}$ are constant. Therefore, the coefficients $\mathcal{H}_\mathcal{M}, \mathcal{K}_\mathcal{M}, \mathcal{H}_\mathcal{L}, \mathcal{K}_\mathcal{L}$ must also be constant.

It follows from the expressions (33), (34), (35), (36) that for functions G_1, G_2, G_3 partial sums of Fourier series have the form:

$$\begin{gathered}
G_1 = S_1^1\sin\gamma + S_1^3\sin3\gamma + S_1^5\sin5\gamma + S_1^7\sin7\gamma,\\
G_2 = C_2^0 + C_2^2\cos2\gamma + C_2^4\cos4\gamma + C_2^6\cos6\gamma + C_2^8\cos8\gamma,\\
G_3 = C_3^1\cos\gamma + C_3^3\cos3\gamma + C_3^5\cos5\gamma + C_3^7\cos7\gamma.
\end{gathered} \tag{38}$$

8. Constructing a Periodic Solution

Passing to constructing an approximate analytical solution of the system (37), it should be noted that the amplitude value of the function $G_1'(\gamma)$ is much greater than the amplitude value of the function $G_2'(\gamma)$. Numerical calculation (Fig. 3) shows that the ratio of the amplitudes of these functions is approximately equal to 26.

Therefore, the system (37) can be simplified by replacing it with a system with skew-symmetric matrices for both angles and their derivatives:

$$\begin{pmatrix}\varphi''\\ \theta''\\ \psi''\end{pmatrix} + \begin{pmatrix}\mathcal{H}_\mathcal{M}+\mathcal{H}_\mathcal{L} & -G_3 & G_2\\ G_3 & \mathcal{H}_\mathcal{L} & -G_1\\ -G_2 & G_1 & \mathcal{H}_\mathcal{M}\end{pmatrix}\begin{pmatrix}\varphi'\\ \theta'\\ \psi'\end{pmatrix} + \begin{pmatrix}\mathcal{K}_\mathcal{M}+\mathcal{K}_\mathcal{L} & 0 & 0\\ 0 & \mathcal{K}_\mathcal{L} & -G_1'\\ 0 & G_1' & \mathcal{K}_\mathcal{M}\end{pmatrix}\begin{pmatrix}\varphi\\ \theta\\ \psi\end{pmatrix} - \mathcal{K}_\mathcal{M}\begin{pmatrix}0\\ \varphi\psi\\ 0\end{pmatrix} + \begin{pmatrix}G_1'\\ G_2'\\ G_3'\end{pmatrix} = 0. \tag{39}$$

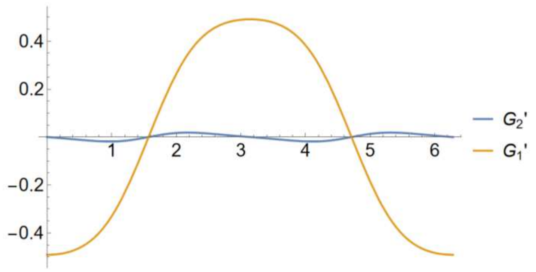

Fig. 3. Functions $G_1'(\gamma)$ and $G_2'(\gamma)$.

We will look for a periodic solution of the system (39), which describes the steady-state oscillations of the satellite $Cxyz$ axes in the WBT coordinate system.

To construct steady-state oscillations, including in nonlinear differential systems, the harmonic balance method is usually used, especially in situations where, due to the required accuracy, it is sufficient to construct a periodic solution in the first approximation, taking into account harmonic oscillations of only the main frequency. When using this method to construct a solution which takes into account higher-order harmonics, that is, in the form of partial sums of trigonometric series, we meet overcoming significant technical difficulties of solving systems of nonlinear algebraic equations to find the coefficients of these trigonometric series.

In this article, to address these challenges, we present an alternative approach to constructing periodic solutions that incorporates higher harmonics for a non-autonomous differential system of the sixth order. The method of generating solutions is based on introducing auxiliary imaginary variables used to convert the initial six-order non-autonomous system of differential equations to an autonomous eight-order system of differential equations. This transformation simplifies the process of analytically finding the coefficients of a trigonometric series, as it reduces to sequential transformations using recursive formulas.

To derive an approximate analytical solution for the system (39), we introduce two auxiliary variables $y_1 = e^{i\gamma}, y_2 = e^{-i\gamma}$. This is sufficient to autonomize the system (39) via using the equalities $\frac{y_1+y_2}{2} = cos\,\gamma$ and $\frac{y_1-y_2}{2i} = sin\,\gamma$ to eliminate its explicit dependence on the variable γ. With this, we introduce two additional differential equations $y_1' = iy_1,\ y_2' = -iy_2$, which variables y_1 and y_2 satisfy [26].

Let us expand the functions in the equations according to the powers of their arguments, i.e., by powers of variables $\varphi, \varphi', \theta, \theta', \psi, \psi', y_1, y_2$. This transforms the system (39) into the form:

$$\begin{aligned}
&\begin{pmatrix}\varphi''\\ \theta''\\ \psi''\end{pmatrix} + \begin{pmatrix}\mathcal{H}_{\mathcal{L}}+\mathcal{H}_{\mathcal{M}} & 0 & C_2^0\\ 0 & \mathcal{H}_{\mathcal{L}} & 0\\ -C_2^0 & 0 & \mathcal{H}_{\mathcal{M}}\end{pmatrix}\begin{pmatrix}\varphi'\\ \theta'\\ \psi'\end{pmatrix} + \begin{pmatrix}\mathcal{K}_{\mathcal{L}}+\mathcal{K}_{\mathcal{M}} & 0 & 0\\ 0 & \mathcal{K}_{\mathcal{L}} & 0\\ 0 & 0 & \mathcal{K}_{\mathcal{M}}\end{pmatrix}\begin{pmatrix}\varphi\\ \theta\\ \psi\end{pmatrix}\\
&+\varphi'\left[y_1\begin{pmatrix}0\\ C_3^1/2\\ 0\end{pmatrix} + y_1^2\begin{pmatrix}0\\ 0\\ -C_2^2/2\end{pmatrix} + y_1^3\begin{pmatrix}0\\ C_3^3/2\\ 0\end{pmatrix} + y_1^4\begin{pmatrix}0\\ 0\\ -C_2^4/2\end{pmatrix} + y_1^5\begin{pmatrix}0\\ C_3^5/2\\ 0\end{pmatrix} + y_1^6\begin{pmatrix}0\\ 0\\ -C_2^6/2\end{pmatrix}\right]+\\
&+\varphi'\left[y_2\begin{pmatrix}0\\ C_3^1/2\\ 0\end{pmatrix} + y_2^2\begin{pmatrix}0\\ 0\\ -C_2^2/2\end{pmatrix} + y_2^3\begin{pmatrix}0\\ C_3^3/2\\ 0\end{pmatrix} + y_2^4\begin{pmatrix}0\\ 0\\ -C_2^4/2\end{pmatrix} + y_2^5\begin{pmatrix}0\\ C_3^5/2\\ 0\end{pmatrix} + y_2^6\begin{pmatrix}0\\ 0\\ -C_2^6/2\end{pmatrix}\right]+\\
&+\theta'\left[y_1\begin{pmatrix}-C_3^1/2\\ 0\\ S_1^1/2i\end{pmatrix} + y_1^3\begin{pmatrix}-C_3^3/2\\ 0\\ S_1^3/2i\end{pmatrix} + y_1^5\begin{pmatrix}-C_3^5/2\\ 0\\ S_1^5/2i\end{pmatrix}\right] + \theta'\left[y_2\begin{pmatrix}-C_3^1/2\\ 0\\ -S_1^1/2i\end{pmatrix} + y_2^3\begin{pmatrix}-C_3^3/2\\ 0\\ -S_1^3/2i\end{pmatrix} + y_2^5\begin{pmatrix}-C_3^5/2\\ 0\\ -S_1^5/2i\end{pmatrix}\right]+\\
&+\psi'\left[y_1\begin{pmatrix}0\\ -S_1^1/2i\\ 0\end{pmatrix} + y_1^2\begin{pmatrix}C_2^2/2\\ 0\\ 0\end{pmatrix} + y_1^3\begin{pmatrix}0\\ -S_1^3/2i\\ 0\end{pmatrix} + y_1^4\begin{pmatrix}C_2^4/2\\ 0\\ 0\end{pmatrix} + y_1^5\begin{pmatrix}0\\ -S_1^5/2i\\ 0\end{pmatrix} + y_1^6\begin{pmatrix}C_2^6/2\\ 0\\ 0\end{pmatrix}\right]+\\
&+\psi'\left[y_2\begin{pmatrix}0\\ S_1^1/2i\\ 0\end{pmatrix} + y_2^2\begin{pmatrix}C_2^2/2\\ 0\\ 0\end{pmatrix} + y_2^3\begin{pmatrix}0\\ S_1^3/2i\\ 0\end{pmatrix} + y_2^4\begin{pmatrix}C_2^4/2\\ 0\\ 0\end{pmatrix} + y_2^5\begin{pmatrix}0\\ S_1^5/2i\\ 0\end{pmatrix} + y_2^6\begin{pmatrix}C_2^6/2\\ 0\\ 0\end{pmatrix}\right]+\\
&+\theta\left[y_1\begin{pmatrix}0\\ 0\\ S_1^1/2\end{pmatrix} + y_1^3\begin{pmatrix}0\\ 0\\ 3S_1^3/2\end{pmatrix} + y_1^5\begin{pmatrix}0\\ 0\\ 5S_1^5/2\end{pmatrix}\right] + \theta\left[y_2\begin{pmatrix}0\\ 0\\ S_1^1/2\end{pmatrix} + y_2^3\begin{pmatrix}0\\ 0\\ 3S_1^3/2\end{pmatrix} + y_2^5\begin{pmatrix}0\\ 0\\ 5S_1^5/2\end{pmatrix}\right]+\\
&+\psi\left[y_1\begin{pmatrix}0\\ -S_1^1/2\\ 0\end{pmatrix} + y_1^3\begin{pmatrix}0\\ -3S_1^3/2\\ 0\end{pmatrix} + y_1^5\begin{pmatrix}0\\ -5S_1^5/2\\ 0\end{pmatrix}\right] + \psi\left[y_2\begin{pmatrix}0\\ -S_1^1/2\\ 0\end{pmatrix} + y_2^3\begin{pmatrix}0\\ -3S_1^3/2\\ 0\end{pmatrix} + y_2^5\begin{pmatrix}0\\ -5S_1^5/2\\ 0\end{pmatrix}\right] - \varphi\psi\begin{pmatrix}0\\ \mathcal{K}_{\mathcal{M}}\\ 0\end{pmatrix}+\\
&+y_1\begin{pmatrix}S_1^1/2\\ 0\\ -C_3^1/2i\end{pmatrix} + y_1^2\begin{pmatrix}0\\ -C_2^2/i\\ 0\end{pmatrix} + y_1^3\begin{pmatrix}3S_1^3/2\\ 0\\ -3C_3^3/2i\end{pmatrix} + y_1^4\begin{pmatrix}0\\ -2C_2^4/i\\ 0\end{pmatrix} + y_1^5\begin{pmatrix}5S_1^5/2\\ 0\\ -5C_3^5/2i\end{pmatrix} + y_1^6\begin{pmatrix}0\\ -3C_2^6/i\\ 0\end{pmatrix}\\
&+y_2\begin{pmatrix}S_1^1/2\\ 0\\ C_2^1/2i\end{pmatrix} + y_2^2\begin{pmatrix}0\\ C_2^2/i\\ 0\end{pmatrix} + y_2^3\begin{pmatrix}3S_1^3/2\\ 0\\ 3C_3^3/2i\end{pmatrix} + y_2^4\begin{pmatrix}0\\ 2C_2^4/i\\ 0\end{pmatrix} + y_2^5\begin{pmatrix}5S_1^5/2\\ 0\\ 5C_3^5/2i\end{pmatrix} + y_2^6\begin{pmatrix}0\\ 3C_2^6/i\\ 0\end{pmatrix} = 0.
\end{aligned} \tag{40}$$

The expansion of the left side of the system (40) is the six-order system, and we will look for a solution in the form of a partial sum of the Fourier series of the 5th order:

$$\begin{aligned}
(\varphi(\gamma), \theta(\gamma), \psi(\gamma))^\top &= A_{10}y_1 + A_{01}y_2 + A_{20}y_1^2 + A_{11}y_1y_2 + A_{02}y_2^2 + A_{30}y_1^3A + A_{21}y_1^2y_2 + A_{12}y_1y_2^2 + A_{03}y_2^3\\
&+A_{40}y_1^4 + A_{31}y_1^3y_2 + A_{22}y_1^2y_2^2 + A_{13}y_1y_2^3 + A_{04}y_2^4 + A_{50}y_1^5 + A_{41}y_1^4y_2 + A_{32}y_1^3y_2^2 + A_{23}y_1^2y_2^3 + A_{14}y_1y_2^4 + A_{05}y_2^5,
\end{aligned} \tag{41}$$

where $A_{ik} = (\Phi_{ik}, \Theta_{ik}, \Psi_{ik})^\top$ are the constant vector coefficients. The solution is a real function, so $A_{ik} = \overline{A}_{ki}$, and:

$$\begin{aligned}
(\varphi(\gamma), \theta(\gamma), \psi(\gamma))^\top = 2\big(&-\sin(\gamma)\,\mathrm{Im}(A_{10}+A_{21}+A_{32}) - \sin(2\gamma)\,\mathrm{Im}(A_{20}+A_{31}) - \sin(3\gamma)\,\mathrm{Im}(A_{30}+A_{41}) - \sin(4\gamma)\,\mathrm{Im}(A_{40})\\
&- \sin(5\gamma)\,\mathrm{Im}(A_{50}) + \cos(2\gamma)\,\mathrm{Re}(A_{20}+A_{31}) + \cos(\gamma)\,\mathrm{Re}(A_{10}+A_{21}+A_{32}) + \cos(4\gamma)\,\mathrm{Re}(A_{40})\\
&+ \cos(3\gamma)\,\mathrm{Re}(A_{30}+A_{41}) + \cos(5\gamma)\,\mathrm{Re}(A_{50})\big) + \mathrm{Re}(A_{11}+A_{22})
\end{aligned}$$

When searching for these coefficients, we will not use the equation $y_1y_2 = 1$, but will consider y_1 and y_2 as independent variables. When substituting Eq. (41) into Eq. (40) we first equate the coefficients at the first powers of y_1 and y_2. Thus, we obtain the following systems of linear algebraic equations with respect to $\Phi_{10}, \Theta_{10}, \Psi_{10}$ and $\Phi_{01}, \Theta_{01}, \Psi_{01}$:

$$\begin{aligned}
&\begin{pmatrix}\Phi_{10}\{-1 + i(\mathcal{H}_{\mathcal{L}}+\mathcal{H}_{\mathcal{M}}) + (\mathcal{K}_{\mathcal{L}}+\mathcal{K}_{\mathcal{M}})\} + \Psi_{10}C_2^0 i\\ \Theta_{10}\{-1 + i\mathcal{H}_{\mathcal{L}} + \mathcal{K}_{\mathcal{L}}\}\\ \Psi_{10}\{-1 - i\mathcal{H}_{\mathcal{M}} + \mathcal{K}_{\mathcal{M}}\} + iC_2^0\Phi_{10}\end{pmatrix} + \begin{pmatrix}S_1^1/2\\ 0\\ iC_3^1/2\end{pmatrix} = 0,\\
&\begin{pmatrix}\Phi_{01}\{-1 - i(\mathcal{H}_{\mathcal{L}}+\mathcal{H}_{\mathcal{M}}) + (\mathcal{K}_{\mathcal{L}}+\mathcal{K}_{\mathcal{M}})\} - \Psi_{01}C_2^0 i\\ \Theta_{01}\{-1 - i\mathcal{H}_{\mathcal{L}} + \mathcal{K}_{\mathcal{L}}\}\\ \Psi_{01}\{-1 + i\mathcal{H}_{\mathcal{M}} + \mathcal{K}_{\mathcal{M}}\} - iC_2^0\Phi_{10}\end{pmatrix} + \begin{pmatrix}S_1^1/2\\ 0\\ -iC_3^1/2\end{pmatrix} = 0.
\end{aligned} \tag{42}$$

For these systems, we immediately find that $\Theta_{10} = \Theta_{01} = 0$. Then, it is not difficult to find the remaining coefficients:

$$\begin{aligned}
\Phi_{10} &= \frac{C_2^0 C_3^1 + S_1^1(i\mathcal{H}_M + \mathcal{K}_M - 1)}{2\left(C_2^{0\,2} + (\mathcal{H}_M - i(\mathcal{K}_M - 1))(\mathcal{H}_L + \mathcal{H}_M - i(\mathcal{K}_L + \mathcal{K}_M - 1))\right)}, \\
\Psi_{10} &= \frac{iC_2^0 S_1^1 - C_3^1(\mathcal{H}_L + \mathcal{H}_M - i(\mathcal{K}_L + \mathcal{K}_M - 1))}{2\left(C_2^{0\,2} + (\mathcal{H}_M - i(\mathcal{K}_M - 1))(\mathcal{H}_L + \mathcal{H}_M - i(\mathcal{K}_L + \mathcal{K}_M - 1))\right)}, \\
\Phi_{01} &= \frac{C_2^0 C_3^1 + S_1^1(-i\mathcal{H}_M + \mathcal{K}_M - 1)}{2\left(C_2^{0\,2} + (\mathcal{H}_M + i(\mathcal{K}_M - 1))(\mathcal{H}_L + \mathcal{H}_M + i(\mathcal{K}_L + \mathcal{K}_M - 1))\right)}, \\
\Psi_{01} &= -\frac{C_3^1(\mathcal{H}_L + \mathcal{H}_M + i(\mathcal{K}_L + \mathcal{K}_M - 1)) + iC_2^0 S_1^1}{2\left(C_2^{0\,2} + (\mathcal{H}_M + i(\mathcal{K}_M - 1))(\mathcal{H}_L + \mathcal{H}_M + i(\mathcal{K}_L + \mathcal{K}_M - 1))\right)}.
\end{aligned} \tag{43}$$

Now we will equate the coefficients for $y_1^i y_2^k$, where $k + i = 2$, that is, the coefficients of the second order. At the same time, we will immediately take into account that $\Theta_{10} = \Theta_{01} = 0$.

When equating the coefficients at y_1^2, y_2^2 and $y_1 y_2$, we get:

$$\begin{aligned}
&\begin{pmatrix} \Phi_{20}\{-4 + 2i(\mathcal{H}_{\mathcal{L}} + \mathcal{H}_{\mathcal{M}}) + (\mathcal{K}_{\mathcal{L}} + \mathcal{K}_{\mathcal{M}})\} + 2i\Psi_{20}C_2^0 \\ \Theta_{20}\{-4 + 2i\mathcal{H}_{\mathcal{L}} + \mathcal{K}_{\mathcal{L}}\} \\ \Psi_{20}\{-4 + 2i\mathcal{H}_{\mathcal{M}} + \mathcal{K}_{\mathcal{M}}\} - 2iC_2^0\Phi_{20} \end{pmatrix} + \begin{pmatrix} 0 \\ i\Phi_{10}C_3^1/2 - \Psi_{10}S_1^1 + iC_2^2 - \mathcal{K}_M\Phi_{10}\Psi_{10} \\ 0 \end{pmatrix} = 0, \\
&\begin{pmatrix} \Phi_{02}\{-4 - 2i(\mathcal{H}_{\mathcal{L}} + \mathcal{H}_{\mathcal{M}}) + (\mathcal{K}_{\mathcal{L}} + \mathcal{K}_{\mathcal{M}})\} - 2i\Psi_{02}C_2^0 \\ \Theta_{02}\{-4 - 2i\mathcal{H}_{\mathcal{L}} + \mathcal{K}_{\mathcal{L}}\} \\ \Psi_{02}\{-4 - 2i\mathcal{H}_{\mathcal{M}} + \mathcal{K}_{\mathcal{M}}\} + 2i\Phi_{02}C_2^0 \end{pmatrix} + \begin{pmatrix} 0 \\ -i\Phi_{01}C_3^1/2 - \Psi_{01}S_1^1 - iC_2^2 - \mathcal{K}_M\Phi_{01}\Psi_{01} \\ 0 \end{pmatrix} = 0, \\
&\begin{pmatrix} \Phi_{11}(\mathcal{K}_{\mathcal{L}} + \mathcal{K}_{\mathcal{M}}) \\ \Theta_{02}\mathcal{K}_{\mathcal{L}} \\ \Psi_{02}\mathcal{K}_{\mathcal{M}} \end{pmatrix} + \begin{pmatrix} 0 \\ i(\Phi_{10} - \Phi_{01})/2 \\ 0 \end{pmatrix} - \mathcal{K}_M(\Phi_{10}\Psi_{01} + \Phi_{01}\Psi_{10}) = 0.
\end{aligned} \tag{44}$$

It is obvious that the coefficients Φ_{ik} and $\Psi_{ik}, (i + k = 2)$ are equal to zero, and the coefficients $\Theta_{ik}, (i + k = 2)$ are obviously expressed in terms of the coefficients of the first order that have already been found:

$$\begin{aligned}
\Theta_{2,0} &= -\frac{C_3^1\Phi_{10} + 2C_2^2 + 2i\Psi_{10}(\Phi_{10}\mathcal{K}_{\mathcal{M}} + S_1^1)}{4\mathcal{H}_{\mathcal{L}} - 2i(\mathcal{K}_{\mathcal{L}} - 4)}, \\
\Theta_{1,1} &= \frac{2\mathcal{K}_{\mathcal{M}}(\Phi_{10}\Psi_{01} + \Phi_{01}\Psi_{10}) + iC_3^1(\Phi_{01} - \Phi_{10})}{2\mathcal{K}_{\mathcal{L}}}, \\
\Theta_{0,2} &= \frac{-C_3^1\Phi_{01} - 2C_2^2 + 2i\Psi_{01}(\Phi_{01}\mathcal{K}_{\mathcal{M}} + S_1^1)}{4\mathcal{H}_{\mathcal{L}} + 2i(\mathcal{K}_{\mathcal{L}} - 4)}.
\end{aligned} \tag{45}$$

The calculation of all required coefficients is described in detail in Appendix.

9. Computer Simulation and Comparison of Results

Since the goal of our investigation is to construct a periodic solution, it is convenient to represent the result in phase planes (φ, φ'), (θ, θ') and (ψ, ψ'), where periodic solution corresponds to closed phase trajectories. Figures 4 to 6 present these phase planes, numerical solution of precise nonlinear system and analytical solution. For the numerical solution, it is necessary to fix the values of the parameters: $k_{L0} = 0.01893\ \mathrm{N\cdot m}, k_{M0} = 0.01923\ \mathrm{N\cdot m}, h_{L0} = 0.4701\ \mathrm{N\cdot m\cdot s}, h_{M0} = 0.4801\ \mathrm{N\cdot m\cdot s}$. The initial conditions are $\varphi(0), \varphi'(0), \theta(0), \theta'(0), \psi(0), \psi'(0)$.

The analytical solution for the selected parameter values takes the form:

$$\begin{aligned}
\varphi(\gamma) &= 0{,}000659712\sin(\gamma) - 0{,}00155533\sin(3\gamma) - 0{,}000157833\sin(5\gamma) + 0{,}0239189\cos(\gamma) - 0{,}00532652\cos(3\gamma) - 0{,}000111084\cos(5\gamma), \\
\theta(\gamma) &= 0{,}0000147992 + 0{,}004892765\sin(2\gamma) + 0{,}00321607\sin(4\gamma) - 0{,}000852243\cos(2\gamma) + 0{,}000390873\cos(4\gamma), \\
\psi(\gamma) &= -0{,}048987\sin(\gamma) + 0{,}0134634\sin(3\gamma) - 0{,}000247052\sin(5\gamma) + 0{,}00141578\cos(\gamma) - 0{,}00501831\cos(3\gamma) - 0{,}000009.21793\cos(5\gamma).
\end{aligned} \tag{46}$$

The comparison of the constructed analytical solution with the results of computer modeling is represented in terms of average integral deviations. Calculations show the values of integral average deviations $\Delta\Phi, \Delta\Theta, \Delta\Psi$ of the analytical solution from the numerical solution:

$$\Delta\Phi = 0{,}000056, \quad \Delta\Theta = 0{,}000070, \quad \Delta\Psi = 0{,}00014. \tag{47}$$

These results confirm the high accuracy of the constructed approximate analytical solution. It follows from Figs. 4 to 6 that the angles φ, θ and ψ vary respectively within the segments $[-0.024{,}0.024]$, $[-0.008{,}0.007]$, $[-0.07{,}0.07]$.

Now we can compare solutions for another set of the parameters: $k_{L0} = 0.01769\ \mathrm{N\cdot m}$, $k_{M0} = 0.01665\ \mathrm{N\cdot m}, h_{L0} = 0.4330\ \mathrm{N\cdot m\cdot s}, h_{M0} = 0.4285\ \mathrm{N\cdot m\cdot s}$.

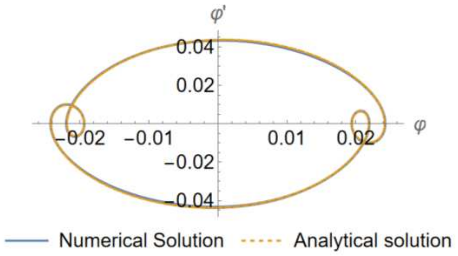

Fig. 4. Phase plane (φ, φ').

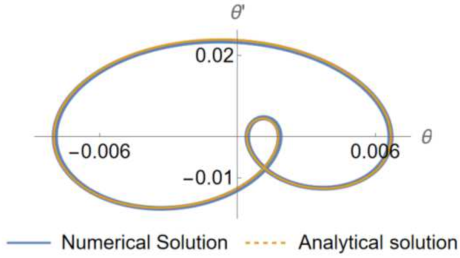

Fig. 5. Phase plane (θ, θ').

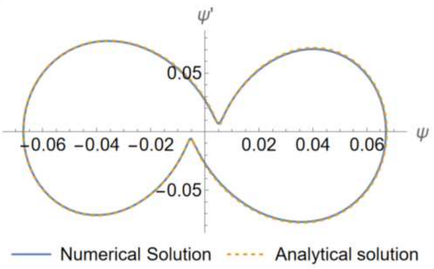

Fig. 6. Phase plane (ψ, ψ').

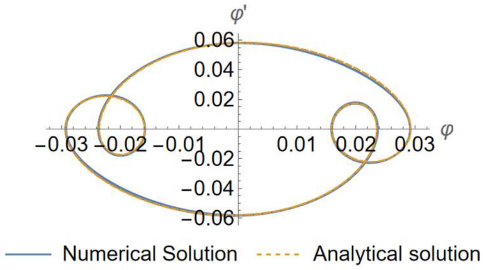

Fig. 7. Phase plane (φ, φ').

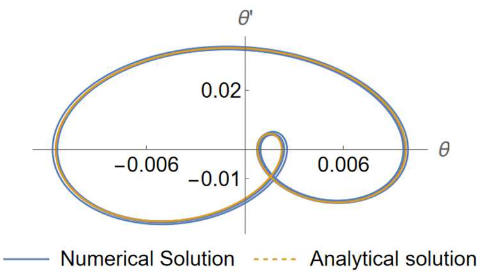

Fig. 8. Phase plane (θ, θ').

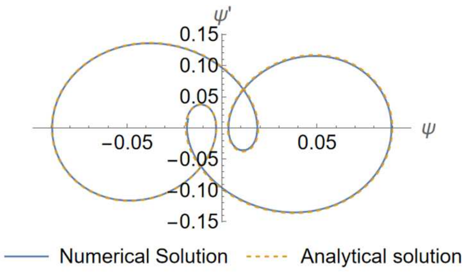

Fig. 9. Phase plane (ψ, ψ').

As we can see, $k_{L0},\ k_{M0}, h_{L0}, h_{M0}$ are now smaller, A is greater, so oscillation amplitude increased. This fully corresponds the physical reality: dissipative and restoring torques decreased and inertia torque did not change. Integral average deviations $\Delta\Phi, \Delta\Theta, \Delta\Psi$ of the analytical solution from the numerical one is as follows:

$$\Delta\Phi = 0.00040775, \quad \Delta\Theta = 0.00109236, \quad \Delta\Psi = 0.000530959. \tag{48}$$

10. Comparison with Harmonic Balance Approximation Method

Now we will try to compare our solution with the one obtained with the harmonic balance method. To use the harmonic balance method, we will seek for the solution in the form:

$$\begin{aligned} \Phi(t) &= \Phi_{21}\sin(t) + \Phi_{22}\sin(2t) + \Phi_{23}\sin(3t) + \Phi_{11}\cos(t) + \Phi_{12}\cos(2t) + \Phi_{13}\cos(3t), \\ \Theta(t) &= \Theta_{21}\sin(t) + \Theta_{22}\sin(2t) + \Theta_{23}\sin(3t) + \Theta_{11}\cos(t) + \Theta_{12}\cos(2t) + \Theta_{13}\cos(3t), \\ \Psi(t) &= \Psi_{21}\sin(t) + \Psi_{22}\sin(2t) + \Psi_{23}\sin(3t) + \Psi_{11}\cos(t) + \Psi_{12}\cos(2t) + \Psi_{13}\cos(3t) \end{aligned} \tag{49}$$

After substituting Eq. (49) in Eq. (39), we get a huge nonlinear system of 18 equations (50), that we cannot separate from each other. The reason why a higher order approximation is not considered is because of the high computational complexity:

$$24.1834\Phi_{11} + 0.630302\Phi_{21} + 0.996421\Psi_{21} + 0.00482853\Psi_{23} + 0.534778\Theta_{22} - 0.532432 = 0, \tag{50a}$$

$$\begin{aligned} 6.34556(&-\Phi_{22}\Psi_{21} - \Phi_{22}\Psi_{23} - \Phi_{12}\Psi_{11} - \Phi_{11}\Psi_{12} - \Phi_{13}\Psi_{12} - \Phi_{12}\Psi_{13} - \Phi_{21}\Psi_{22} - \Phi_{23}\Psi_{22}) - 0.534778\Phi_{22} - 0.273335\Psi_{12} + 11.4923\Theta_{11} \\ &+ 0.311841\Theta_{21} = 0, \end{aligned} \tag{50b}$$

$$-1.00514\Phi_{21} - 0.0116715\Phi_{23} + 11.6911\Psi_{11} + 0.318461\Psi_{21} + 0.273335\Theta_{12} = 0, \tag{50c}$$

$$-0.630302\Phi_{11} + 24.1834\Phi_{21} - 1.00514\Psi_{11} - 0.00389048\Psi_{13} - 0.559627\Theta_{12} = 0, \tag{50d}$$

$$\begin{aligned} 6.34556(&\Phi_{12}\Psi_{21} - \Phi_{12}\Psi_{23} - \Phi_{22}\Psi_{11} + \Phi_{21}\Psi_{12} - \Phi_{23}\Psi_{12} + \Phi_{22}\Psi_{13} - \Phi_{11}\Psi_{22} + \Phi_{13}\Psi_{22}) + 0.559627\Phi_{12} - 0.259097\Psi_{22} - 0.311841\Theta_{11} \\ &+ 11.4923\Theta_{21} = 0, \end{aligned} \tag{50e}$$

$$0.996421\Phi_{11} + 0.0144856\Phi_{13} - 0.318461\Psi_{11} + 11.6911\Psi_{21} + 0.259097\Theta_{22} + 0.547202 = 0, \tag{50f}$$

$$21.1834\Phi_{12} + 1.2606\Phi_{22} + 2.0025\Psi_{22} + 0.267389\Theta_{21} + 0.821325\Theta_{23} = 0, \tag{50g}$$

$$\begin{aligned} 6.34556(\Phi_{21}\Psi_{21} - \Phi_{21}\Psi_{23} - \Phi_{11}\Psi_{11} - \Phi_{13}\Psi_{11} - \Phi_{11}\Psi_{13} - \Phi_{23}\Psi_{21}) - 0.267389\Phi_{21} - 0.821325\Phi_{23} + 0.518194\Psi_{11} - 0.532047\Psi_{13} \\ + 8.49225\Theta_{12} + 0.623682\Theta_{22} = 0, \end{aligned} \tag{50h}$$

$$-2.00062\Phi_{22} + 8.69112\Psi_{12} + 0.636923\Psi_{22} - 0.518194\Theta_{11} + 0.532047\Theta_{13} = 0, \tag{50i}$$

$$-1.2606\Phi_{12} + 21.1834\Phi_{22} - 2.00062\Psi_{12} - 0.279814\Theta_{11} - 0.820283\Theta_{13} = 0, \tag{50j}$$

$$\begin{aligned} 6.34556(-\Phi_{11}\Psi_{21} - \Phi_{11}\Psi_{23} - \Phi_{21}\Psi_{11} - \Phi_{23}\Psi_{11} + \Phi_{21}\Psi_{13} + \Phi_{13}\Psi_{21}) + 0.279814\Phi_{11} + 0.820283\Phi_{13} + 0.54667\Psi_{21} - 0.532817\Psi_{23} \\ - 0.623682\Theta_{12} + 8.49225\Theta_{22} - 0.017438 = 0, \end{aligned} \tag{50k}$$

$$2.0025\Phi_{12} - 0.636923\mathrm{Ps}_{12} + 8.69112\Psi_{22} - 0.54667\Theta_{21} + 0.532817\Theta_{23} = 0, \tag{50l}$$

$$16.1834\Phi_{13} + 1.89091\Phi_{23} + 0.0144856\Psi_{21} + 3.00228\Psi_{23} + 0.54755\Theta_{22} + 0.0427138 = 0, \tag{50m}$$

$$-0.54754\Phi_{22} - 6.34556\Phi_{12}\Psi_{11} + 0.79922\Psi_{12} + 6.34556(-\Phi_{11}\Psi_{12} + \Phi_{22}\Psi_{21} + \Phi_{21}\Psi_{22}) + 3.49225\Theta_{13} + 0.93552\Theta_{23} = 0, \tag{50n}$$

$$-0.00389\Phi_{21} - 3.00239\Phi_{23} + 3.69112\Psi_{13} + 0.95538\Psi_{23} - 0.79922\Theta_{12} = 0, \tag{50o}$$

$$-1.89090\Phi_{13} + 16.18337\Phi_{23} - 0.011671\Psi_{11} - 3.00239\Psi_{13} - 0.54685\Theta_{12} = 0, \tag{50p}$$

$$0.54685\Phi_{12} + 6.34556(-\Phi_{22}\Psi_{11} - \Phi_{21}\Psi_{12} - \Phi_{12}\Psi_{21} - \Phi_{11}\Psi_{22}) + 0.79807\Psi_{22} - 0.93552\Theta_{13} + 3.49225\Theta_{23} = 0, \tag{50q}$$

$$-0.03727 + 0.00482\Phi_{11} + 3.00228\Phi_{13} - 0.95538\Psi_{13} + 3.69112\Psi_{23} - 0.79807\Theta_{22} = 0. \tag{50r}$$

System (50) can be solved numerically, while analytical solution is not a trivial problem. Upon calculating the coefficients for the third approximation and determining the mean integral error, it becomes evident that both methods demonstrate equal accuracy regarding the angles φ and ψ. The phase portraits of the solutions obtained are shown in Figs. 10 to 12.

The mean integral error values for harmonic balance method are as follows:

$$\Delta\Phi = 0.0001818, \quad \Delta\Theta = 0.0026340, \quad \Delta\Psi = 0.0004215. \tag{51}$$

and for complex autonomization method:

$$\Delta\Phi = 0.00021054, \quad \Delta\Theta = 0.0028450, \quad \Delta\Psi = 0.0003451. \tag{52}$$

The deviation in the angle θ for the third approximation in both instances is of the same order as the oscillation amplitude. This is due to the fact that a higher order of approximation, at least the fifth, is required for the theta angle oscillations. The method we employed earlier performs exceptionally well in constructing such an approximation. Meanwhile, the harmonic balance method necessitates resolving a nonlinear system comprising 30 interrelated equations. Solving such a system requires significantly more computational power, which indicates that the method under consideration has a considerable advantage over the classical harmonic balance method. Verification shows that some modifications of the harmonic balance method, which allow avoiding the calculation of solutions to a nonlinear system, do not provide any gain in accuracy either, but require a larger volume of calculations.

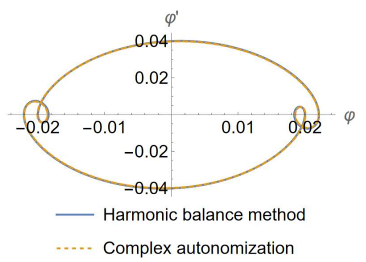

Fig. 10. Phase plane (φ, φ').

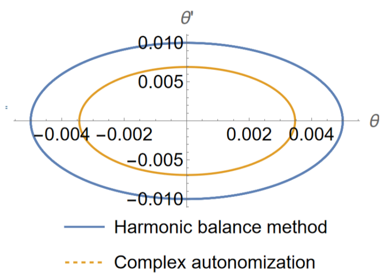

Fig. 11. Phase plane (θ, θ').

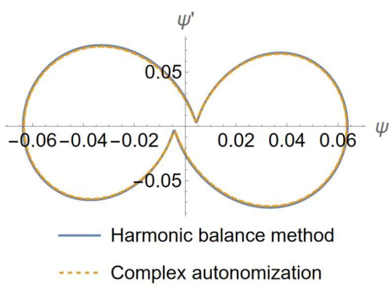

Fig. 12. Phase plane (ψ, ψ').

11. Conclusion

Thus, in the course of the work done, a computer simulation of satellite oscillations in the newly introduced magneto-Lorentz coordinate system in the gravitational and magnetic fields of the Earth was carried out. The study is based on a system of differential equations derived for this purpose, describing the mode of passive electrodynamic attitude control of the satellite.

The method of complex autonomization was used to construct an approximate analytical solution describing the steady-state oscillations of the satellite. By analogy with the harmonic balance method, the periodic solution is constructed as a partial sum of a trigonometric series. Despite this analogy, the proposed method compares favorably with the harmonic balance method for the following reasons. Firstly, unlike the harmonic balance method, which requires solving a system of nonlinear algebraic equations, the proposed method requires solving a much simpler linear system. Secondly, the system of algebraic equations for the coefficients of these series is an easy-solvable system of recurrent equations, which distinguishes it from a similar system of the harmonic balance method, all equations depending on all unknown values at once. Thus, the procedure to find the coefficients of trigonometric series analytically brings no difficulties, since it reduces to elementary transformations (can even be performed manually!) by recurrent formulas.

It is important that an analytical solution describing the steady-state oscillations of the satellite can be obtained in this way with any desired degree of accuracy. In this paper, for example, the expansion of a periodic solution into a trigonometric series up to and including fifth-order harmonics is constructed. A comparison of the constructed approximate analytical solution with the periodic solution obtained numerically shows a high degree of accuracy of the analytical solution in the specified approximation. The practical significance of the method of analytical construction of a periodic solution demonstrated in the work is based on the fact that, at first, it allows to estimate the magnitude of these oscillations with any required degree of accuracy and, therefore, to justify the necessity to take these oscillations into account in the satellite mission. Secondly, the analytical expression of steady-state oscillations allows to make correct adjustments of measurements of on-board instruments and improve the accuracy of information transmitted from the satellite.

Author Contributions

A.S. Lozovoi suggested the experiments, developed the mathematical modeling, conducted computer modeling, and analyzed the numerical results; K.G. Petrov suggested the experiments and conducted computer modeling; A.A. Tikhonov initiated the project, planned the scheme, and examined the theory validation. The manuscript was written through the contribution of all the authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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Not applicable.

Conflict of Interest

The authors declared no potential conflicts of interest concerning the research, authorship, and publication of this article.

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

References

[1] Gorr, G.V., Belokon, T.V., On solutions of the equations of motion of a gyrostat with a variable gyrostatic moment, *Mechanics of Solids*, 56(7), 2021, 1157–1166.

[2] Amer, T.S., Alanazy, A., Elneklawy, A.H., Amer, W.S., El-Kafly, H.F., A novel study on the fourth first integral for the rotatory motion of an impacted charged rigid body by external torques, *Journal of Low Frequency Noise, Vibration and Active Control*, 2025, https://doi.org/10.1177/ 14613484251347080.

[3] Kosov, A.A., On the stability of stationary solutions of the equations of motion of the Goryachev-Sretensky gyrostat, *Mechanics of Solids*, 58, 2023, 1986–1997.

[4] Amer, T.S., Alanazy, A., Elneklawy, A.H., Amer, W.S., El-Kafly, H.F., Asymptotic solutions for the 3D motion of asymmetric charged gyrostatic satellite using Poincare small parameter technique, *Aerospace Science and Technology*, 168, 2026, 110764.

[5] Sarychev, V.A., *Issues of orientation for artificial satellites*, Itogi Nauki Tekh., Ser.: Issled. Kosm. Prostr, vol. 11, 1978, (In Russian).

[6] Beletsky, V.V., Yanshin, A.M., *The Influence of Aerodynamics Forces on Spacecraft Rotation*, Naukova Dumka, Kiev, 1984, (In Russian).

[7] Barinova, E.V., Belokonov, I.V., Timbai, I.A., Technology for designing the angular motion of cubesat nanosatellites with a passive stabilization system, *Gyroscopy and Navigation*, 15(4), 2024, 346-355.

[8] Wertz, J.R., *Spacecraft Attitude Determination and Control*, D. Reidel Publishing Co., Dordrecht, 1985.

[9] Somov Ye., Butyrin S., Somov S., Spatial guidance and control of a space robot during flights in a low-orbit constellation of Earth observation mini-satellites, *Cybernetics and Physics*, 13(1), 2024, 77-81.

[10] Aleksandrov, A.Yu., Antipov, K.A., Platonov, A.V., Tikhonov, A.A., Electrodynamic attitude stabilization of a satellite in the Konig frame, *Nonlinear Dynamics*, 82, 2015, 1493-1505.

[11] Laundal, K.M., Richmond, A.D., Magnetic Coordinate Systems, *Space Science Reviews*, 206, 2017, 27-59.

[12] Beletsky, V.V., Khentov, A.A., *Magnetized Spacecraft Attitude Motion*. Nauka, Moscow, 1985, (In Russian).

[13] Zhou, K., Huang, H., Wang, X., Sun, L., Zhong, R., Magnetic attitude control for Earth-pointing satellites in the presence of gravity gradient, *Aerospace Science and Technology*, 60, 2017, 115-123.

[14] Roldugin, D.S., Extensive numerical simulation of a magnetically actuated satellite rotation around the sun direction with sun sensors data only, *Mathematical Models and Computer Simulations*, 15(5), 2023, 792-801.

[15] Ovchinnikov, M.Y., Roldugin, D.S., A survey on active magnetic attitude control algorithms for small satellites, *Progress in Aerospace Sciences*, 109, 2019, 100546.

[16] Morozov, V.M., Kalenova, V.I., Satellite Control Using Magnetic Moments: Controllability and Stabilization Algorithms, *Cosmic Research*, 58, 2020, 158-166.

[17] Tikhonov, A.A., A method of semipassive attitude stabilization of a spacecraft in the geomagnetic field, *Cosmic Research*, 41(1), 2003, 63-73.
[18] Giri, D.K., Sinha, M., Magneto-Coulombic attitude control of Earth-pointing satellites, *Journal of Guidance, Control, and Dynamics*, 37(6), 2014, 1946-1960.
[19] Giri, D.K., Sinha, M., Three-axis attitude control of Earth-pointing isoinertial magneto-Coulombic satellites, *International Journal of Dynamics and Control*, 5(3), 2017, 644-652.
[20] Prabhat, H., Mukherjee, B.K., Giri, D.K., Sinha, M., Fault-tolerant sliding mode satellite attitude stabilization using magneto-coulombic torquers, *Aerospace Science and Technology*, 121, 2022, 107316.
[21] Antipov, K.A., Tikhonov, A.A., Parametric control in the problem of spacecraft stabilization in the geomagnetic field, *Automation and Remote Control*, 68(8), 2007, 1333-1345.
[22] Tikhonov, A.A., Spasic, D.T., Antipov, K.A., Sablina, M.V., Optimizing the electrodynamical stabilization method for a man-made Earth satellite, *Automation and Remote Control*, 72(9), 2011, 1898-1905.
[23] Klyushin, M.A., Maksimenko, M.V., Tikhonov, A.A., Electrodynamic attitude stabilization of a spacecraft in an elliptical orbit, *Aerospace*, 11, 2024, 956.
[24] Tikhonov, A.A., Natural Magneto-velocity Coordinate System for Satellite Attitude Stabilization: The Concept and Kinematic Analysis, *Journal of Applied and Computational Mechanics*, 7(4), 2021, 2113-2119.
[25] Aleksandrov, A.Yu., Tikhonov, A.A., Natural Magneto-velocity Coordinate System for Satellite Attitude Stabilization: Dynamics and Stability Analysis, *Journal of Applied and Computational Mechanics*, 9(2), 2023, 513-520.
[26] Melnikov, G.I., Tikhonov, A.A., Way of Defining Periodic Movements of Nonautonomous Automatic Systems, *Automation and Remote Control*, 7, 1970, 5–14.

Appendix

A.1. Calculation of g_i Functions

Representations for absolute values of $\vec{B}$, $\vec{T}$ and $\vec{W}$ vectors have the form:

$$|\vec{B}| = \delta\sqrt{\alpha^2+\beta^2\sin^2\gamma+4\beta^2\cos^2\gamma} = \delta\sqrt{\alpha^2+\beta^2+3\beta^2\cos^2 v},$$

$$|\vec{T}| = \sqrt{v_c^2 B_\zeta^2 + v_c^2 B_\eta^2} = v_c\delta\sqrt{\alpha^2+4\beta^2\cos^2\gamma}, \tag{A.1}$$

$$|\vec{W}| = |\vec{B}||\vec{T}|.$$

The same absolute values in the introduced notation:

$$|\vec{B}| = \Pi\delta\sqrt{1+\Delta_1\cos^2\gamma},$$

$$|\vec{T}| = v_c\delta\alpha\sqrt{1+\Delta_2\cos^2\gamma}, \tag{A.2}$$

$$|\vec{W}| = \Pi\delta^2 v_c\alpha\sqrt{1+\Delta_1\cos^2\gamma}\,\sqrt{1+\Delta_2\cos^2\gamma}.$$

Now we can derive the formulas for b_{ij}:

$$b_{11} = \frac{\alpha}{\Pi}\sqrt{\frac{1+\Delta_2\cos^2\gamma}{1+\Delta_1\cos^2\gamma}} \qquad b_{21} = \frac{\beta}{\Pi}\frac{\sin\gamma}{\sqrt{1+\Delta_1\cos^2\gamma}} \qquad b_{31} = 0$$

$$b_{12} = -\frac{\beta}{\Pi}\frac{\sin\gamma}{\sqrt{1+\Delta_1\cos^2\gamma}\,\sqrt{1+\Delta_2\cos^2\gamma}} \qquad b_{22} = \frac{\alpha}{\Pi}\frac{1}{\sqrt{1+\Delta_1\cos^2\gamma}} \qquad b_{32} = -\frac{2\beta}{\alpha}\frac{\cos\gamma}{\sqrt{1+\Delta_2\cos^2\gamma}} \tag{A.3}$$

$$b_{13} = -\frac{\beta^2}{\alpha\Pi}\frac{\sin 2\gamma}{\sqrt{1+\Delta_1\cos^2\gamma}\,\sqrt{1+\Delta_2\cos^2\gamma}} \qquad b_{23} = \frac{2\beta}{\Pi}\frac{\cos\gamma}{\sqrt{1+\Delta_1\cos^2\gamma}} \qquad b_{33} = \frac{1}{\sqrt{1+\Delta_2\cos^2\gamma}},$$

Taking into account that $dB_\xi/d\gamma = B_\zeta/2,\ dB_\zeta/d\gamma = -2B_\xi,\ dB_\eta/d\gamma = 0$, we get:

$$|\vec{B}|' = \frac{\mathrm{d}}{\mathrm{d}\gamma}\sqrt{B_\xi^2+B_\eta^2+B_\zeta^2} = \frac{B_\xi B_\xi' + B_\eta B_\eta' + B_\zeta B_\zeta'}{\sqrt{B_\xi^2+B_\eta^2+B_\zeta^2}} = -\frac{3}{2}\frac{B_\xi B_\zeta}{|\vec{B}|} = -\frac{3\delta^2\beta^2\sin 2\gamma}{2|\vec{B}|} \tag{A.4}$$

$$|\vec{T}|' = v_c\frac{d}{d\gamma}\sqrt{B_\eta^2+B_\zeta^2} = v_c\frac{B_\eta B_\eta' + B_\zeta B_\zeta'}{\sqrt{B_\eta^2+B_\zeta^2}} = -2v_c^2\frac{B_\xi B_\zeta}{|\vec{T}|} = -2v_c^2\frac{\delta^2\beta^2\sin 2\gamma}{|\vec{T}|}, \tag{A.5}$$

Here and further, the (…)' sign denotes derivative with respect to γ. Before finding g_i, let's write the expressions for $\omega_{b_0 i}$. Taking into account that $\dot{b}_{ij} = b_{ij}'(\omega_0-\omega_E) = b_{ij}'\Delta\omega$, we get the following:

$$\begin{aligned}\omega_{b_0 1}/\Delta\omega &= b_{31}\dot{b}_{21} + b_{32}\dot{b}_{22} + b_{33}\dot{b}_{23} = 0 - v_c\frac{B_\zeta}{|\vec{T}|}\left(\frac{B_\eta}{|\vec{B}|}\right)' + v_c\frac{B_\eta}{|\vec{T}|}\left(\frac{B_\zeta}{|\vec{B}|}\right)' = -v_c\frac{B_\zeta}{|\vec{T}|}\left(\frac{B_\eta'|\vec{B}| - B_\eta|\vec{B}|'}{|\vec{B}|^2}\right) + v_c\frac{B_\eta}{|\vec{T}|}\left(\frac{B_\zeta'|\vec{B}| - B_\zeta|\vec{B}|'}{|\vec{B}|^2}\right)\\ &= \frac{v_c}{|\vec{B}|^2|\vec{T}|}\left(-B_\eta' B_\zeta|\vec{B}| + B_\zeta B_\eta|\vec{B}|' + B_\eta B_\zeta'|\vec{B}| - B_\eta B_\zeta|\vec{B}|'\right) = \frac{v_c B_\eta B_\zeta'}{|\vec{B}||\vec{T}|} = -2\frac{v_c B_\xi B_\eta}{|\vec{B}||\vec{T}|}\\ &= -\frac{2\beta}{\Pi}\frac{\sin\gamma}{\sqrt{1+\Delta_1\cos^2\gamma}\,\sqrt{1+\Delta_2\cos^2\gamma}}\end{aligned} \tag{A.6}$$

$$\begin{aligned}\omega_{b_0 2}/\Delta\omega &= b_{11}\dot{b}_{31} + b_{12}\dot{b}_{32} + b_{13}\dot{b}_{33} = 0 - \frac{v_c B_\eta B_\xi}{|\vec{T}||\vec{B}|}\left(-\frac{v_c B_\zeta}{|\vec{T}|}\right)' - \frac{v_c B_\xi B_\zeta}{|\vec{T}||\vec{B}|}\left(\frac{v_c B_\eta}{|\vec{T}|}\right)' \\ &= \frac{v_c^2}{|\vec{T}||\vec{B}|}\left[B_\eta B_\xi\left(\frac{B_\zeta'|\vec{T}| - B_\zeta|\vec{T}|'}{|\vec{T}|^2}\right) - B_\xi B_\zeta\left(\frac{B_\eta'|\vec{T}| - B_\eta|\vec{T}|'}{|\vec{T}|^2}\right)\right] = \\ &= \frac{v_c^2}{|\vec{T}|^3|\vec{B}|}\left(B_\eta B_\xi B_\zeta'|\vec{T}| - B_\eta B_\xi B_\zeta|\vec{T}|' - B_\xi B_\zeta B_\eta'|\vec{T}| + B_\xi B_\zeta B_\eta|\vec{T}|'\right) = \frac{v_c^2 B_\eta B_\xi B_\zeta'}{|\vec{T}|^2|\vec{B}|} = -2\frac{v_c^2 B_\eta B_\xi^2}{|\vec{T}|^2|\vec{B}|} \\ &= -\frac{2\beta^2}{\Pi\alpha}\frac{\sin^2\gamma}{\sqrt{1+\Delta_1\cos^2\gamma}\,(1+\Delta_2\cos^2\gamma)}\end{aligned} \tag{A.7}$$

$$\begin{aligned}\omega_{b_0 3}/\Delta\omega &= b_{21}\dot{b}_{11} + b_{22}\dot{b}_{12} + b_{23}\dot{b}_{13} = \frac{B_\xi}{|\vec{B}|}\left(\frac{v_c(B_\eta^2 + B_\zeta^2)}{|\vec{B}||\vec{T}|}\right)' + \frac{B_\eta}{|\vec{B}|}\left(-\frac{v_c B_\eta B_\xi}{|\vec{B}||\vec{T}|}\right)' + \frac{B_\zeta}{|\vec{B}|}\left(-\frac{v_c B_\xi B_\zeta}{|\vec{B}||\vec{T}|}\right)' = \\ &= \frac{v_c}{|\vec{B}|}\left[B_\xi\left(\frac{(2B_\eta B_\eta' + 2B_\zeta B_\zeta')|\vec{B}||\vec{T}| - (|\vec{B}||\vec{T}|)'(B_\eta^2 + B_\zeta^2)}{|\vec{B}|^2|\vec{T}|^2}\right) - B_\eta\left(\frac{(B_\eta' B_\xi + B_\eta B_\xi')|\vec{B}||\vec{T}| - (|\vec{B}||\vec{T}|)' B_\eta B_\xi}{|\vec{B}|^2|\vec{T}|^2}\right)\right. \\ &\left. - B_\zeta\left(\frac{(B_\xi' B_\zeta + B_\xi B_\zeta')|\vec{B}||\vec{T}| - (|\vec{B}||\vec{T}|)' B_\xi B_\zeta}{|\vec{B}|^2|\vec{T}|^2}\right)\right] = \\ &= \frac{v_c}{|\vec{B}|^3|\vec{T}|^2}\left[2B_\xi B_\zeta B_\zeta'|\vec{B}||\vec{T}| - B_\xi B_\eta^2(|\vec{B}||\vec{T}|)' - B_\xi B_\zeta^2(|\vec{B}||\vec{T}|)' - B_\eta^2 B_\xi'|\vec{B}||\vec{T}| + B_\xi B_\eta^2(|\vec{B}||\vec{T}|)' - B_\zeta^2 B_\xi'|\vec{B}||\vec{T}|\right. \\ &\left. - B_\xi B_\zeta' B_\zeta|\vec{B}||\vec{T}| + B_\zeta^2 B_\xi(|\vec{B}||\vec{T}|)'\right] = \frac{v_c}{|\vec{B}|^2|\vec{T}|}\left[B_\xi B_\zeta B_\zeta' - B_\xi'(B_\eta^2 + B_\zeta^2)\right] = -\frac{v_c}{|\vec{B}|^2|\vec{T}|}\left[2B_\xi^2 B_\zeta + B_\xi'(B_\eta^2 + B_\zeta^2)\right] = \\ &= -\frac{v_c}{|\vec{B}|^2|\vec{T}|}\left[4\delta^3\beta^3\sin^2\gamma\cos\gamma + \delta^3\beta\cos\gamma\,(\alpha^2 + 4\beta^2\cos^2\gamma)\right] = -\frac{v_c\delta^3\beta\cos\gamma}{|\vec{B}|^2|\vec{T}|}\left[4\beta^2\sin^2\gamma + \alpha^2 + 4\beta^2\cos^2\gamma\right] \\ &= -\frac{v_c\delta^3\beta\cos\gamma}{|\vec{B}|^2|\vec{T}|}\left[4\beta^2 + \alpha^2\right] = -\frac{\beta(4\beta^2 + \alpha^2)}{\alpha\Pi^2}\frac{\cos\gamma}{(1+\Delta_1\cos^2\gamma)\sqrt{1+\Delta_2\cos^2\gamma}},\end{aligned} \tag{A.8}$$

Now, we can derive the expressions for g_i:

$$\begin{aligned}g_1 &= -\frac{2\beta}{\Pi}\frac{\sin\gamma}{\sqrt{1+\Delta_1\cos^2\gamma}\,\sqrt{1+\Delta_2\cos^2\gamma}}\Delta\omega - \frac{\beta}{\Pi}\frac{\sin\gamma}{\sqrt{1+\Delta_1\cos^2\gamma}\,\sqrt{1+\Delta_2\cos^2\gamma}}\omega_0 \\ &= -(3\omega_0 - 2\omega_3)\frac{\beta}{\Pi}\frac{\sin\gamma}{\sqrt{1+\Delta_1\cos^2\gamma}\,\sqrt{1+\Delta_2\cos^2\gamma}}\end{aligned} \tag{A.9}$$

$$g_2 = -\frac{2\beta^2}{\Pi\alpha}\frac{\sin^2\gamma}{\sqrt{1+\Delta_1\cos^2\gamma}\,(1+\Delta_2\cos^2\gamma)}\Delta\omega + \frac{\alpha}{\Pi}\frac{1}{\sqrt{1+\Delta_1\cos^2\gamma}}\omega_0 \tag{A.10}$$

$$g_3 = -\frac{\beta(4\beta^2 + \alpha^2)}{\alpha\Pi^2}\frac{\cos\gamma}{(1+\Delta_1\cos^2\gamma)\sqrt{1+\Delta_2\cos^2\gamma}}\Delta\omega - \frac{2\beta}{\alpha}\frac{\cos\gamma}{\sqrt{1+\Delta_2\cos^2\gamma}}\omega_0, \tag{A.11}$$

A.2. Systems of Equations for Finding Coefficients of the Third and Higher Orders

Further let's find coefficients of the third order, by equating expressions with the same powers after substituting Eq. (24) in Eq. (23). Equating expressions with y_1^3:

$$\begin{pmatrix}\Phi_{30}\{-9 + 3i(\mathcal{H}_\mathcal{L} + \mathcal{H}_\mathcal{M}) + (\mathcal{K}_\mathcal{L} + \mathcal{K}_\mathcal{M})\} + 3i\Psi_{30}C_2^0 \\ \Theta_{30}\{-9 + 3i\mathcal{H}_\mathcal{L} + \mathcal{K}_\mathcal{L}\} \\ \Psi_{30}\{-9 + 3i\mathcal{H}_\mathcal{M} + \mathcal{K}_\mathcal{M}\} - 3iC_2^0\Phi_{30}\end{pmatrix} + i\Phi_{10}\begin{pmatrix}0 \\ 0 \\ -C_2^2/2\end{pmatrix} + 2i\Theta_{20}\begin{pmatrix}-C_3^1/2 \\ 0 \\ S_1^1/2i\end{pmatrix} + i\Psi_{10}\begin{pmatrix}C_2^2/2 \\ 0 \\ B_M/2\end{pmatrix} + \Theta_{20}\begin{pmatrix}0 \\ 0 \\ S_1^1/2\end{pmatrix} + \begin{pmatrix}3S_1^3/2 \\ 0 \\ i3C_3^3/2\end{pmatrix} = 0, \tag{A.12}$$

Equating expressions with y_2^3:

$$\begin{pmatrix}\Phi_{03}\{-9 - 3i(\mathcal{H}_\mathcal{L} + \mathcal{H}_\mathcal{M}) + (\mathcal{K}_\mathcal{L} + \mathcal{K}_\mathcal{M})\} - 3i\Psi_{03}C_2^0 \\ \Theta_{03}\{-9 - 3i\mathcal{H}_\mathcal{L} + \mathcal{K}_\mathcal{L}\} \\ \Psi_{03}\{-9 - 3i\mathcal{H}_\mathcal{M} + \mathcal{K}_\mathcal{M}\} + 3iC_2^0\Phi_{03}\end{pmatrix} - i\Phi_{01}\begin{pmatrix}0 \\ 0 \\ -C_2^2/2\end{pmatrix} - 2i\Theta_{02}\begin{pmatrix}-C_3^1/2 \\ 0 \\ -S_1^1/2i\end{pmatrix} - i\Psi_{01}\begin{pmatrix}C_2^2/2 \\ 0 \\ 0\end{pmatrix} + \Theta_{02}\begin{pmatrix}0 \\ 0 \\ S_1^1/2\end{pmatrix} + \begin{pmatrix}3S_1^3/2 \\ 0 \\ -i3C_3^3/2\end{pmatrix} = 0. \tag{A.13}$$

Equating expressions with $y_1^2 y_2$:

$$\begin{pmatrix}\Phi_{21}\{-1 + i(\mathcal{H}_\mathcal{L} + \mathcal{H}_\mathcal{M}) + (\mathcal{K}_\mathcal{L} + \mathcal{K}_\mathcal{M})\} + i\Psi_{21}C_2^0 \\ \Theta_{21}\{-1 + i\mathcal{H}_\mathcal{L} + \mathcal{K}_\mathcal{L}\} \\ \Psi_{21}\{-1 + i\mathcal{H}_\mathcal{M} + \mathcal{K}_\mathcal{M}\} - iC_2^0\Phi_{21}\end{pmatrix} - i\Phi_{01}\begin{pmatrix}0 \\ 0 \\ -C_2^2/2\end{pmatrix} - i\Psi_{01}\begin{pmatrix}C_2^2/2 \\ 0 \\ 0\end{pmatrix} + 2i\Theta_{20}\begin{pmatrix}-C_3^1/2 \\ 0 \\ -S_1^1/2i\end{pmatrix} + \Theta_{11}\begin{pmatrix}0 \\ 0 \\ S_1^1/2\end{pmatrix} = 0. \tag{A.14}$$

Equating expressions with $y_2^2 y_1$:

$$\begin{pmatrix} \Phi_{12}\{-1-i(\mathcal{H}_\mathcal{L}+A_M)+(\mathcal{K}_\mathcal{L}+\mathcal{K}_\mathcal{M})\}-i\Psi_{12}C_2^0 \\ \Theta_{12}\{-1-i\mathcal{H}_\mathcal{L}+\mathcal{K}_\mathcal{L}\} \\ \Psi_{12}\{9-iA_M+\mathcal{K}_\mathcal{M}\}+iC_2^0\Phi_{12} \end{pmatrix} + i\Phi_{10}\begin{pmatrix}0\\0\\-C_2^2/2\end{pmatrix} + i\Psi_{10}\begin{pmatrix}C_2^2/2\\0\\0\end{pmatrix} - 2i\Theta_{02}\begin{pmatrix}-C_3^1/2\\0\\S_1^1/2i\end{pmatrix} + \Theta_{11}\begin{pmatrix}0\\0\\S_1^1/2\end{pmatrix} = 0. \tag{A.15}$$

When solving these four linear systems, we find all the coefficients $\Phi_{ik}, \Theta_{ik}, \Psi_{ik}$ for $i+k=3$. However, it can be noted that the Θ-coefficients of the third order are equal to zero.

This pattern will be observed: Θ coefficients of odd orders will always be zero, and Φ, Ψ coefficients will be zero in cases where they have an even order. Let's continue in finding the coefficients of the fourth order ($i+k=4$). Equating expressions with y_1^4:

$$\begin{gathered}\begin{pmatrix} \Phi_{40}\{-16+4i(\mathcal{H}_\mathcal{L}+\mathcal{H}_\mathcal{M})+(\mathcal{K}_\mathcal{L}+\mathcal{K}_\mathcal{M})\}+4i\Psi_{40}C_2^0 \\ \Theta_{40}\{-16+4i\mathcal{H}_\mathcal{L}+\mathcal{K}_\mathcal{L}\} \\ \Psi_{40}\{-16+4i\mathcal{H}_\mathcal{M}+\mathcal{K}_\mathcal{M}\}-4iC_2^0\Phi_{40} \end{pmatrix} + 3i\Phi_{30}\begin{pmatrix}0\\C_3^1/2\\0\end{pmatrix} + i\Phi_{10}\begin{pmatrix}0\\C_3^3/2\\0\end{pmatrix} + 3i\Psi_{30}\begin{pmatrix}0\\-S_1^1/2i\\0\end{pmatrix} + i\Psi_{10}\begin{pmatrix}0\\-S_1^3/2i\\0\end{pmatrix} \\ + \Psi_{30}\begin{pmatrix}0\\-S_1^1/2\\0\end{pmatrix} + \Psi_{10}\begin{pmatrix}0\\-3S_1^3/2\\0\end{pmatrix} + \begin{pmatrix}0\\2iC_2^4\\0\end{pmatrix} - \mathcal{K}_\mathcal{M}\begin{pmatrix}0\\ \Phi_{20}\Psi_{20}+\Phi_{30}\Psi_{10}+\Phi_{10}\Psi_{30}\\0\end{pmatrix} = 0.\end{gathered} \tag{A.16}$$

Equating expressions with y_2^4:

$$\begin{gathered}\begin{pmatrix} \Phi_{04}\{-16-4i(\mathcal{H}_\mathcal{L}+\mathcal{H}_\mathcal{M})+(\mathcal{K}_\mathcal{L}+\mathcal{K}_\mathcal{M})\}-4i\Psi_{04}C_2^0 \\ \Theta_{04}\{-16-4i\mathcal{H}_\mathcal{L}+\mathcal{K}_\mathcal{L}\} \\ \Psi_{04}\{-16+4i\mathcal{H}_\mathcal{M}+\mathcal{K}_\mathcal{M}\}+4iC_2^0\Phi_{04} \end{pmatrix} - 3i\Phi_{03}\begin{pmatrix}0\\C_3^1/2\\0\end{pmatrix} - i\Phi_{01}\begin{pmatrix}0\\C_3^3/2\\0\end{pmatrix} - 3i\Psi_{03}\begin{pmatrix}0\\S_1^1/2i\\0\end{pmatrix} - i\Psi_{01}\begin{pmatrix}0\\S_1^3/2i\\0\end{pmatrix} \\ + \Psi_{03}\begin{pmatrix}0\\-S_1^1/2\\0\end{pmatrix} + \Psi_{01}\begin{pmatrix}0\\-3S_1^3/2\\0\end{pmatrix} + \begin{pmatrix}0\\-2iC_2^4\\0\end{pmatrix} - \mathcal{K}_\mathcal{M}\begin{pmatrix}0\\ \Phi_{02}\Psi_{02}+\Phi_{03}\Psi_{01}+\Phi_{01}\Psi_{03}\\0\end{pmatrix} = 0.\end{gathered} \tag{A.17}$$

Equating expressions with $y_1^3 y_2$:

$$\begin{gathered}\begin{pmatrix} \Phi_{31}\{-4+2i(\mathcal{H}_\mathcal{L}+\mathcal{H}_\mathcal{M})+(\mathcal{K}_\mathcal{L}+\mathcal{K}_\mathcal{M})\}+2i\Psi_{31}C_2^0 \\ \Theta_{31}\{-4+2i\mathcal{H}_\mathcal{L}+\mathcal{K}_\mathcal{L}\} \\ \Psi_{31}\{-4+2i\mathcal{H}_\mathcal{M}+\mathcal{K}_\mathcal{M}\}-2iC_2^0\Phi_{31} \end{pmatrix} + i\Phi_{21}\begin{pmatrix}0\\C_3^1/2\\0\end{pmatrix} - i\Phi_{01}\begin{pmatrix}0\\C_3^3/2\\0\end{pmatrix} + 3i\Phi_{30}\begin{pmatrix}0\\C_3^1/2\\0\end{pmatrix} + i\Psi_{21}\begin{pmatrix}0\\-S_1^1/2i\\0\end{pmatrix} \\ + 3i\Psi_{30}\begin{pmatrix}0\\S_1^1/2i\\0\end{pmatrix} + \Psi_{21}\begin{pmatrix}0\\-S_1^1/2\\0\end{pmatrix} + \Psi_{30}\begin{pmatrix}0\\-S_1^1/2\\0\end{pmatrix} + \Psi_{01}\begin{pmatrix}0\\-3S_1^3/2\\0\end{pmatrix} - i\Psi_{01}\begin{pmatrix}0\\-S_1^3/2i\\0\end{pmatrix} \\ - \mathcal{K}_\mathcal{M}\begin{pmatrix}0\\ \Phi_{30}\Psi_{01}+\Phi_{21}\Psi_{10}+\Phi_{10}\Psi_{21}+\Phi_{01}\Psi_{30}\\0\end{pmatrix} = 0.\end{gathered} \tag{A.18}$$

Equating expressions with $y_2^3 y_1$:

$$\begin{gathered}\begin{pmatrix} \Phi_{13}\{-4-2i(\mathcal{H}_\mathcal{L}+\mathcal{H}_\mathcal{M})+(\mathcal{K}_\mathcal{L}+\mathcal{K}_\mathcal{M})\}-2i\Psi_{13}C_2^0 \\ \Theta_{13}\{-4-2i\mathcal{H}_\mathcal{L}+\mathcal{K}_\mathcal{L}\} \\ \Psi_{13}\{-4-2i\mathcal{H}_\mathcal{M}+\mathcal{K}_\mathcal{M}\}+2iC_2^0\Phi_{13} \end{pmatrix} - i\Phi_{12}\begin{pmatrix}0\\C_3^1/2\\0\end{pmatrix} + i\Phi_{10}\begin{pmatrix}0\\C_3^3/2\\0\end{pmatrix} - 3i\Phi_{03}\begin{pmatrix}0\\C_3^1/2\\0\end{pmatrix} + i\Psi_{12}\begin{pmatrix}0\\-S_1^1/2i\\0\end{pmatrix} \\ + 3i\Psi_{03}\begin{pmatrix}0\\S_1^1/2i\\0\end{pmatrix} + \Psi_{12}\begin{pmatrix}0\\-S_1^1/2\\0\end{pmatrix} + \Psi_{03}\begin{pmatrix}0\\-S_1^1/2\\0\end{pmatrix} + \Psi_{10}\begin{pmatrix}0\\-3S_1^3/2\\0\end{pmatrix} - i\Psi_{10}\begin{pmatrix}0\\-S_1^3/2i\\0\end{pmatrix} \\ - \mathcal{K}_\mathcal{M}\begin{pmatrix}0\\ \Phi_{03}\Psi_{10}+\Phi_{12}\Psi_{01}+\Phi_{01}\Psi_{12}+\Phi_{10}\Psi_{03}\\0\end{pmatrix} = 0.\end{gathered} \tag{A.19}$$

Equating expressions with $y_1^2 y_2^2$:

$$\begin{gathered}\begin{pmatrix} \Phi_{22}(\mathcal{K}_\mathcal{L}+\mathcal{K}_\mathcal{M}) \\ \Theta_{22}\mathcal{K}_\mathcal{L} \\ \Psi_{22}\mathcal{K}_\mathcal{M} \end{pmatrix} - i\Phi_{12}\begin{pmatrix}0\\C_3^1/2\\0\end{pmatrix} - i\Psi_{12}\begin{pmatrix}0\\-S_1^1/2i\\0\end{pmatrix} + \Theta_{02}\begin{pmatrix}0\\Q_L/2\\0\end{pmatrix} + \Psi_{12}\begin{pmatrix}0\\-S_1^1/2\\0\end{pmatrix} + i\Phi_{21}\begin{pmatrix}0\\C_3^1/2\\0\end{pmatrix} + i\Psi_{21}\begin{pmatrix}0\\S_1^1/2i\\0\end{pmatrix} \\ + \Psi_{21}\begin{pmatrix}0\\-S_1^1/2\\0\end{pmatrix} - \mathcal{K}_\mathcal{M}\begin{pmatrix}0\\ \Phi_{12}\Psi_{01}+\Phi_{10}\Psi_{03}+\Phi_{03}\Psi_{10}+\Phi_{01}\Psi_{12}\\0\end{pmatrix} = 0.\end{gathered} \tag{A.20}$$

and coefficients of the fifth order; Equating expressions with y_1^5:

$$\begin{gathered}\begin{pmatrix} \Phi_{50}\{-25+5i(\mathcal{H}_\mathcal{L}+A_M)+(\mathcal{K}_\mathcal{L}+\mathcal{K}_\mathcal{M})\}+5i\Psi_{50}C_2^0 \\ \Theta_{50}\{-25+5i\mathcal{H}_\mathcal{L}+\mathcal{K}_\mathcal{L}\} \\ \Psi_{50}\{-25+5iA_M+\mathcal{K}_\mathcal{M}\}-5iC_2^0\Phi_{50} \end{pmatrix} + 3i\Phi_{30}\begin{pmatrix}0\\0\\-C_2^2/2\end{pmatrix} + i\Phi_{10}\begin{pmatrix}0\\0\\-C_2^4/2\end{pmatrix} + 4i\Theta_{04}\begin{pmatrix}-C_3^1/2\\0\\S_1^1/2i\end{pmatrix} \\ + 2i\Theta_{20}\begin{pmatrix}-C_3^3/2\\0\\S_1^3/2i\end{pmatrix} + 3i\Psi_{30}\begin{pmatrix}C_2^2/2\\0\\0\end{pmatrix} + i\Psi_{10}\begin{pmatrix}C_2^4/2\\0\\0\end{pmatrix} + \Theta_{40}\begin{pmatrix}0\\0\\S_1^1/2\end{pmatrix} + \Theta_{20}\begin{pmatrix}0\\0\\3S_1^3/2\end{pmatrix} + \begin{pmatrix}5S_1^5/2\\0\\i5C_3^5/2\end{pmatrix} = 0.\end{gathered} \tag{A.21}$$

Equating expressions with y_2^5:

$$\begin{gathered}\begin{pmatrix} \Phi_{05}\{-25-5i(\mathcal{H}_\mathcal{L}+\mathcal{H}_\mathcal{M})+(\mathcal{K}_\mathcal{L}+\mathcal{K}_\mathcal{M})\}-5i\Psi_{05}C_2^0 \\ \Theta_{05}\{-25-5i\mathcal{H}_\mathcal{L}+\mathcal{K}_\mathcal{L}\} \\ \Psi_{05}\{-25-5i\mathcal{H}_\mathcal{M}+\mathcal{K}_\mathcal{M}\}+5iC_2^0\Phi_{05} \end{pmatrix} - 3i\Phi_{03}\begin{pmatrix}0\\0\\-C_2^2/2\end{pmatrix} - i\Phi_{01}\begin{pmatrix}0\\0\\-C_2^4/2\end{pmatrix} + 4i\Theta_{04}\begin{pmatrix}C_3^1/2\\0\\S_1^1/2i\end{pmatrix} + 2i\Theta_{02}\begin{pmatrix}C_3^3/2\\0\\S_1^3/2i\end{pmatrix} \\ - 3i\Psi_{03}\begin{pmatrix}C_2^2/2\\0\\0\end{pmatrix} - i\Psi_{01}\begin{pmatrix}C_2^4/2\\0\\0\end{pmatrix} + \Theta_{04}\begin{pmatrix}0\\0\\S_1^1/2\end{pmatrix} + \Theta_{02}\begin{pmatrix}0\\0\\3S_1^3/2\end{pmatrix} + \begin{pmatrix}5S_1^5/2\\0\\-i5C_3^5/2\end{pmatrix} = 0.\end{gathered} \tag{A.22}$$

Equating expressions with $y_1^4 y_2$:

$$\begin{aligned}&\begin{pmatrix}\Phi_{41}\{-9+3i(\mathcal{H}_{\mathcal{L}}+\mathcal{H}_{\mathcal{M}})+(\mathcal{K}_{\mathcal{L}}+\mathcal{K}_{\mathcal{M}})\}+3i\Psi_{41}C_2^0\\ \Theta_{41}\{-9+3i\mathcal{H}_{\mathcal{L}}+\mathcal{K}_{\mathcal{L}}\}\\ \Psi_{41}\{-9+3i\mathcal{H}_{\mathcal{M}}+\mathcal{K}_{\mathcal{M}}\}-3iC_2^0\Phi_{41}\end{pmatrix}+i\Phi_{21}\begin{pmatrix}0\\0\\-C_2^2/2\end{pmatrix}-i\Phi_{01}\begin{pmatrix}0\\0\\-C_2^4/2\end{pmatrix}+2i\Theta_{31}\begin{pmatrix}-C_3^1/2\\0\\S_1^1/2i\end{pmatrix}\\ &\quad+4i\Theta_{40}\begin{pmatrix}-C_3^1/2\\0\\-S_1^1/2i\end{pmatrix}+i\Psi_{21}\begin{pmatrix}C_2^2/2\\0\\0\end{pmatrix}-i\Psi_{01}\begin{pmatrix}C_2^4/2\\0\\0\end{pmatrix}+\Theta_{31}\begin{pmatrix}0\\0\\S_1^1/2\end{pmatrix}+\Theta_{40}\begin{pmatrix}0\\0\\S_1^1/2\end{pmatrix}+\Theta_{11}\begin{pmatrix}0\\0\\3S_1^3/2\end{pmatrix}=0.\end{aligned}\tag{A.23}$$

Equating expressions with $y_2^4 y_1$:

$$\begin{aligned}&\begin{pmatrix}\Phi_{14}\{-9-3i(\mathcal{H}_{\mathcal{L}}+A_M)+(\mathcal{K}_{\mathcal{L}}+\mathcal{K}_{\mathcal{M}})\}-3i\Psi_{14}C_2^0\\ \Theta_{14}\{-9-3i\mathcal{H}_{\mathcal{L}}+\mathcal{K}_{\mathcal{L}}\}\\ \Psi_{14}\{-9-3iA_M+\mathcal{K}_{\mathcal{M}}\}+3iC_2^0\Phi_{41}\end{pmatrix}-i\Phi_{12}\begin{pmatrix}0\\0\\-C_2^2/2\end{pmatrix}+i\Phi_{10}\begin{pmatrix}0\\0\\-C_2^4/2\end{pmatrix}+2i\Theta_{13}\begin{pmatrix}C_3^1/2\\0\\S_1^1/2i\end{pmatrix}+4i\Theta_{04}\begin{pmatrix}C_3^1/2\\0\\-S_1^1/2i\end{pmatrix}\\ &\quad-i\Psi_{12}\begin{pmatrix}C_2^2/2\\0\\0\end{pmatrix}+i\Psi_{10}\begin{pmatrix}C_2^4/2\\0\\0\end{pmatrix}+\Theta_{13}\begin{pmatrix}0\\0\\S_1^1/2\end{pmatrix}+\Theta_{04}\begin{pmatrix}0\\0\\S_1^1/2\end{pmatrix}+\Theta_{11}\begin{pmatrix}0\\0\\3S_1^3/2\end{pmatrix}=0.\end{aligned}\tag{A.24}$$

Equating expressions with $y_1^3 y_2^2$:

$$\begin{aligned}&\begin{pmatrix}\Phi_{32}\{-1+i(\mathcal{H}_{\mathcal{L}}+\mathcal{H}_{\mathcal{M}})+(\mathcal{K}_{\mathcal{L}}+\mathcal{K}_{\mathcal{M}})\}+i\Psi_{32}C_2^0\\ \Theta_{32}\{-1+i\mathcal{H}_{\mathcal{L}}+\mathcal{K}_{\mathcal{L}}\}\\ \Psi_{32}\{-1+i\mathcal{H}_{\mathcal{M}}+\mathcal{K}_{\mathcal{M}}\}-iC_2^0\Phi_{32}\end{pmatrix}-i\Phi_{12}\begin{pmatrix}0\\0\\-C_2^2/2\end{pmatrix}+3i\Phi_{30}\begin{pmatrix}0\\0\\-C_2^2/2\end{pmatrix}-2i\Theta_{02}\begin{pmatrix}-C_3^3/2\\0\\S_1^3/2i\end{pmatrix}+2i\Theta_{31}\begin{pmatrix}-C_3^1\\0\\-S_1^1\end{pmatrix}\\ &\quad+\Theta_{22}\begin{pmatrix}0\\0\\S_1^1/2\end{pmatrix}+\Theta_{31}\begin{pmatrix}0\\0\\S_1^1/2\end{pmatrix}+\Theta_{02}\begin{pmatrix}0\\0\\3S_1^3/2\end{pmatrix}-i\Psi_{12}\begin{pmatrix}C_2^2/2\\0\\0\end{pmatrix}+3i\Psi_{30}\begin{pmatrix}C_2^2\\0\\0\end{pmatrix}=0.\end{aligned}\tag{A.25}$$

Equating expressions with $y_1^2 y_2^3$:

$$\begin{aligned}&\begin{pmatrix}\Phi_{23}\{-1-i(\mathcal{H}_{\mathcal{L}}+\mathcal{H}_{\mathcal{M}})+(\mathcal{K}_{\mathcal{L}}+\mathcal{K}_{\mathcal{M}})\}-i\Psi_{23}C_2^0\\ \Theta_{23}\{-1-i\mathcal{H}_{\mathcal{L}}+\mathcal{K}_{\mathcal{L}}\}\\ \Psi_{23}\{-1-i\mathcal{H}_{\mathcal{M}}+\mathcal{K}_{\mathcal{M}}\}=iC_2^0\Phi_{23}\end{pmatrix}+i\Phi_{21}\begin{pmatrix}0\\0\\-C_2^2/2\end{pmatrix}-3i\Phi_{03}\begin{pmatrix}0\\0\\-C_2^2/2\end{pmatrix}-2i\Theta_{20}\begin{pmatrix}C_3^3/2\\0\\S_1^3/2i\end{pmatrix}-2i\Theta_{13}\begin{pmatrix}-C_3^1\\0\\-S_1^1\end{pmatrix}\\ &\quad+\Theta_{22}\begin{pmatrix}0\\0\\S_1^1/2\end{pmatrix}+\Theta_{13}\begin{pmatrix}0\\0\\S_1^1/2\end{pmatrix}+\Theta_{20}\begin{pmatrix}0\\0\\3S_1^3/2\end{pmatrix}+i\Psi_{21}\begin{pmatrix}C_2^2/2\\0\\0\end{pmatrix}-3i\Psi_{03}\begin{pmatrix}C_2^2\\0\\0\end{pmatrix}=0.\end{aligned}\tag{A.26}$$

By sequentially solving these systems, it is possible to explicitly find expressions for the desired coefficients. The analytical expressions are quite complicated; therefore, they are not given here.

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