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Research Paper

Large Time Behaviour of Solutions to the Rotating Navier-Stokes Equations on the β-plane

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Abstract. In this paper, we investigate the two-dimensional rotating Navier–Stokes equations on a periodic β-plane, where the Coriolis parameter varying as βy, in the strip domain $\Omega = \mathbb{T} \times \mathbb{R}$. Utilizing a horizontal space decomposition method, we demonstrate that the oscillatory component of the velocity v decays over time. Furthermore, we establish the global existence of strong solutions for this rotating system with horizontal dissipation only. In addition, we prove that the oscillatory part of the velocity decays exponentially to zero in the H^1 norm as $t \to \infty$.

Keywords: Rotating Navier-Stokes equations; Global strong solutions; Horizontal space decomposition; Exponential decay.

1. Introduction

On a rotating sphere, the strength of the Coriolis effect depends on latitude and is quantified by the Coriolis parameter F (see [1, 2]). In a suitable coordinate system, the Coriolis parameter F is expressed as $F_0 + \beta y$, where F_0 represents the forcing term on the velocity, and β is the Coriolis coefficient. In this work, we investigate the large-time stability of the system when $F_0 = 0$. More explicitly, we consider the following rotating Navier–Stokes equations under β-plane approximation in the strip domain $\Omega = \mathbb{T} \times \mathbb{R}$, where the interval $\mathbb{T}$ is taken to be periodic, namely, $\mathbb{T} := [0,1]$:

$$\begin{cases} v_t + v \cdot \nabla v + \nabla P = \nu \Delta v - \beta y J v, \ (x,t) \in \Omega \times \mathbb{R}^+ \\ \mathrm{div} v = 0, \end{cases} \tag{1}$$

On a rotating sphere, where $v = (v_1, v_2)^T$ refers to the fluid velocity, P stands for the pressure and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the rotation matrix. The parameter β belongs to real-valued parameters that determines the intensity of the Coriolis force, while ν denotes the kinematic viscosity. The term $\beta y J v$ stems from the spatial variation in the frame's rotation and can be interpreted as a linear approximation of a portion of a rotating sphere.

The first equation in Eq. (1) represents the balance of linear momentum, where the advection, pressure gradient, viscous diffusion, and rotational effects are all taken into account, while the second equation enforces the incompressibility condition, ensuring the conservation of mass. The Coriolis term $\beta y J v$ appears when Newton's laws are formulated in a rotating frame associated with the Earth, and it encodes the impact of planetary rotation on the flow. Importantly, this rotational effect induces a strong anisotropy in the flow: it suppresses variations along the axis of rotation, enhances horizontal coherence and favors the emergence of large-scale structures. As a consequence, the rotating incompressible Navier–Stokes system serves as a fundamental model in geophysical fluid dynamics, capturing phenomena such as geostrophic balance, Rossby wave propagation, and the formation of zonal jets (see, e.g., [3, 4]).

For $\beta = 0$, the system (2) simplifies to the standard 2D Navier–Stokes equations. Returning to the rotating Navier–Stokes system (2), basic physical reasoning and numerical results [5, 6] indicate that a rotation varying as βy encourages the solution to become more zonally aligned (a zonal flow is one that is independent of x). This implies that, with the addition of differential rotation, a small number of fairly distinct flow regimes emerge. Coherent vortices weaken and eventually disappear as the strength of the rotational effect increases, accompanied by increased anisotropy. To verify this experimental inference, the vorticity field $\omega(x,y,t)$ is decomposed as $\omega(x,y,t) = \bar{\omega}(x,y,t) + \tilde{\omega}(x,y,t)$, where $\bar{\omega}$ represents the zonal portion, derived through averaging ω over x. It has been shown in [7, 8] that the non-zonal component of the flow diminishes as $t \to \infty$.

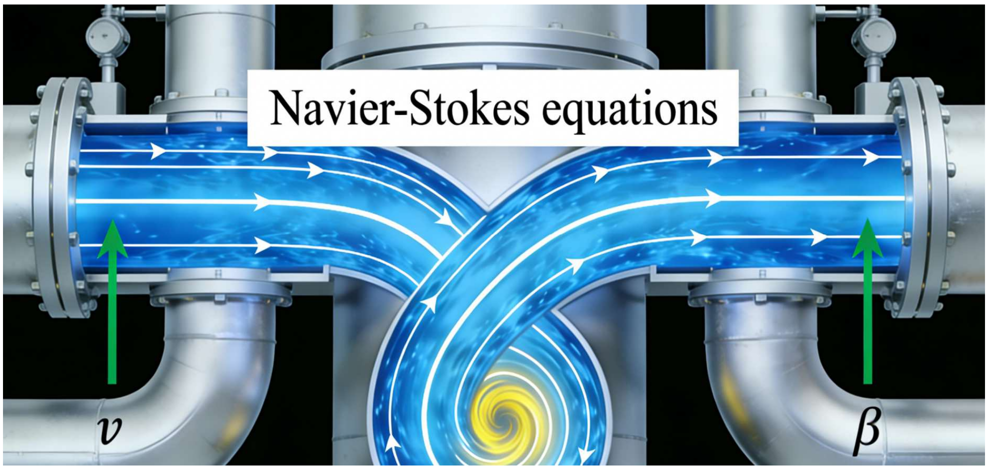

Fig. 1. Rotating Navier-Stokes Systems on the β-plane.

Extending the results from [7], Miyajima and Wirosoetisno [9] established estimates for the determining modes and nodes associated with the degrees of freedom in the rotating Navier–Stokes equations. However, these results heavily depend on the following symmetric initial conditions:

$$v_1(x,-y,t) = v_1(x,y,t), \quad -v_2(x,-y,t) = v_2(x,y,t).$$

In reality, these symmetric initial conditions are difficult to achieve. Therefore, in the absence of such unrealistic initial values, and in order to highlight the anisotropy induced by enhanced rotational effects, we study the following rotating system with only horizontal viscosity (see Fig. 1):

$$\begin{cases} v_t + v\cdot\nabla v + \nabla P = \nu\partial_1^2 v - \beta y J v, \ (x,t)\in\Omega\times\mathbb{R}^+ \\ \mathrm{div} v = 0. \end{cases} \tag{2}$$

The anisotropic rotating Navier–Stokes system (2) naturally arises in geophysical and industrial contexts in which rotational effects are significant and dissipation acts predominantly in certain spatial directions. A classic example occurs in Ekman boundary layers [1, 10], where the competition between viscosity and the Coriolis force produces a thin layer near solid boundaries. In such regions, the motion varies over a vertical scale that is significantly less than the horizontal one, leading to a pronounced disparity between horizontal and vertical diffusion. Similar anisotropic diffusion mechanisms also appear in centrifuge-driven flows and rotating machinery applications [11]. In these scenarios, the horizontal viscosity plays a leading role while the vertical viscosity becomes negligible, making the simplified form Eq. (2) an appropriate and physically relevant reduced model.

The first goal of the present study is to demonstrate the existence of global solutions to Eq. (2). By using the method of horizontal spatial decomposition (see Lemma 2.1), we split v into two parts: $v := v^0 + v^{\neq}$, where v^0 denotes the horizontal average, namely, $v_0 := (\mathbb{P}_0 v)(y) = \int_{\mathbb{T}} v(x,y)dx$ and $v^{\neq}$ is the corresponding oscillation part, namely, $v_{\neq} := \mathbb{P}_{\neq} v = v - \mathbb{P}_0 v$. Since the horizontal average vanishes, $v^{\neq}$ satisfies a strong version of Poincare's inequality (see Eq. (5)). Using these properties, we prove the stability estimate Eq. (3), which ensures the global existence of strong solutions (see Theorem 1.1). Additionally, by decomposing Eq. (2) into its horizontal average and oscillatory parts, it is showed that $v^{\neq}$ tends to zero at an exponentially rate in H^1 as $t\to\infty$ (see Theorem 1.2). For further results on the stability problem of the other fluids systems with partial dissipation, one may refer to [12–14] for the Boussinesq equations, one may refer to [15, 16] for the micropolar system, and to [17–20] for the MHD system.

Before presenting our results, we define a few simplified notations. Let $\partial_1 := \frac{\partial}{\partial x}$, $\partial_2 := \frac{\partial}{\partial y}$, $\Omega := \mathbb{T}\times\mathbb{R}$, $\int \cdot\ dX := \int_\Omega \cdot\ dxdy$, $L^p := L^p(\Omega) := W^{0,p}(\Omega)$ for $1 < p \leqslant \infty$, $H^k := H^k(\Omega)$, $(u_{in}, \omega_{in}) := (u,\omega)|_{t=0}$. We write $a \lesssim b$ to mean $a \leqslant cb$, where k is a non-negative integer and c is a generic positive constant.

The first main result is presented below.

Theorem 1.1. Suppose that the initial data $v_{in}\in H^2$ and satisfies $\mathrm{div} v_{in} = 0$. Then there exists a sufficiently small constant $\varepsilon > 0$ such that, if:

$$\|v_{in}\|_{H^2} \leqslant \varepsilon,$$

the system (2) admits a (global) unique strong solution v, which satisfies, for any $t > 0$,

$$\|v\|_{H^2}^2 + \int_0^t \|\partial_1 v\|_{H^2}^2 d\tau \lesssim \varepsilon. \tag{3}$$

Remark 1.1. The statements of Theorem 1.1 continue to be valid for systems with full viscosity, namely, for system (1.1), or when $\beta = 0$.

Furthermore, we establish that $\|v^{\neq}\|_{H^1}$ decays exponentially in time.

Theorem 1.2. Let v be the global strong solution to Eqs. (2) established in Theorem 1.1. Then the oscillatory part $v^{\neq}$ decays exponentially in time. More precisely, for any $t > 0$,

$$\|v^{\neq}\|_{H^1}^2 + \int_0^t e^{c(\tau-t)}\|\partial_1 v\|_{H^1}^2 d\tau \lesssim e^{-ct}\|v_{in}^{\neq}\|_{H^1}^2, \tag{4}$$

where $c > 0$ is a constant.

The remainder of the paper is structured as follows. In Section 2, we present a few elementary facts, and provide the demonstration of Theorem 1.1. In Section 3, we complete the demonstration of Theorem 1.2.

2. Proof of Theorem 1.1

The purpose of this section is to the proof of Theorem 1.1, where employing a horizontal spatial decomposition technique. This decomposition gives rise to several key properties, which are summarized in Lemma 2.1.

Lemma 2.1. [21]
(1) For the integers $s \geqslant 0$, the average component f^0 and the oscillatory component $f^{\neq}$ are orthogonal in H^s, and:

$$\|f\|_{H^s}^2 = \|f^{\neq}\|_{H^s}^2 + \|f^0\|_{H^s}^2.$$

Moreover, one has:

$$\|f\|_{H^s}^2 \geqslant \|f^{\neq}\|_{H^s}^2 \quad and \quad \|f\|_{H^s}^2 \geqslant \|f^0\|_{H^s}^2.$$

(2) The derivatives commute with both the mean component f^0 and the oscillatory componment $f^{\neq}$,

$$(\partial_k f)^{\neq} = \partial_k f^{\neq}, \qquad (\partial_k f)^0 = \partial_k f^0,$$

for $k = 1{,}2$.
(3) If f satisfies $\mathrm{d}ivf = 0$, the following holds:

$$\mathrm{d}ivf^0 = 0 \quad and \quad \mathrm{d}ivf^{\neq} = 0.$$

Lemma 2.2. Poincare's inequality [17]: Assume $f \in H^1$ with the corresponding oscillatory part $f^{\neq}$. Then $f^{\neq} \in H^1$ and satisfies the estimate:

$$\|f^{\neq}\|_{L^2} \lesssim \|\partial_1 f^{\neq}\|_{L^2}. \tag{5}$$

Lemma 2.3. [17] Suppose that all the right-hand sides are bounded. Then, we have:

$$\int f^{\neq} gh\mathrm{d}X \lesssim \|\partial_1 f^{\neq}\|_{L^2} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}. \tag{6}$$

Proposition 2.1. Assume $v_{in} \in H^2$. Then, we have the following estimate:

$$\|v\|_{H^2}^2 + \int_0^T \|\partial_1 v\|_{H^2}^2 \mathrm{d}t \lesssim \|v_{in}\|_{H^2}^2 + sup_{0 \leqslant t \leqslant T} \|v\|_{H^2} \int_0^T \|\partial_1 v\|_{H^2}^2 \mathrm{d}t. \tag{7}$$

Proof. By forming the L^2 inner product of the first equation of Eq. (2) with v, we perform the integration by parts and $(2)_2$ to obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L^2}^2 + \nu\|\partial_1 v\|_{L^2}^2 = 0, \tag{8}$$

where we have used the facts that:

$$\begin{aligned}\int (v \cdot \nabla v + \nabla P) \cdot v\mathrm{d}X &= \int v \cdot \nabla \left(P + \frac{1}{2}|v|^2\right) \mathrm{d}X \\ &= -\int \left(\frac{1}{2}|v|^2 + P\right) \mathrm{d}iv v\mathrm{d}X = 0,\end{aligned}$$

and,

$$\int \beta yJv \cdot v\mathrm{d}X = \int \beta y(-v_2 v_1 + v_1 v_2)\mathrm{d}X = 0.$$

To get the H^1 estimate of v, applying the vorticity operator $curl$ to Eq. (2) and using $curl\nabla P = 0$, for $\omega := curlv = \partial_1 v_2 - \partial_2 v_1$, we get:

$$\begin{cases}\omega_t + v \cdot \nabla\omega = \nu\partial_1^2\omega - \beta v_2, \\ \mathrm{d}ivv = 0.\end{cases} \tag{9}$$

Multiplying $(9)_1$ by ω in L^2, by using integration by parts, we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^2}^2 + \nu\|\partial_1\omega\|_{L^2}^2 = -\beta\int v_2\,\omega\mathrm{d}X, \tag{10}$$

where we used:

$$\int v \cdot \nabla\omega\omega\mathrm{d}X = \frac{1}{2}\int v \cdot \nabla|\omega|^2\mathrm{d}X = -\frac{1}{2}\int \mathrm{d}iv\,v|\omega|^2\mathrm{d}X = 0.$$

For the linear term appearing on the right-hand of Eq. (10), invoking the decomposition $v = v^0 + v^{\neq}$ and using Eq. $(9)_2$, Young's inequality and Poincare's inequality in Eq. (5), we get:

$$\begin{aligned}\beta\int v_2\,\omega\mathrm{d}X &= \beta\int (-\partial_2 v_1 + \partial_1 v_2)\,v_2\mathrm{d}X \\ &= \beta\int (v_2\partial_1 v_2 + v_1\partial_2 v_2)\,\mathrm{d}X \\ &= \beta\int \left(v_2\partial_1\left(v_2^0 + v_2^{\neq}\right) - v_1\partial_1\left(v_1^0 + v_1^{\neq}\right)\right)\mathrm{d}X \\ &= \beta\int \left(v_2^{\neq}\partial_1 v_2^{\neq} - v_1^{\neq}\partial_1 v_1^{\neq}\right)\mathrm{d}X\end{aligned} \tag{11}$$

$$
\begin{aligned}
&\lesssim \beta\left(\|v_2^{\neq}\|_{L^2}\|\partial_1 v_2^{\neq}\|_{L^2}+\|v_1^{\neq}\|_{L^2}\|\partial_1 v_1^{\neq}\|_{L^2}\right)\\
&\lesssim \beta\left(\left\|\partial_1 v_2^{\neq}\|_{L^2}\right\|\partial_1^2 v_2^{\neq}\|_{L^2}+\left\|\partial_1 v_1^{\neq}\|_{L^2}\right\|\partial_1^2 v_1^{\neq}\|_{L^2}\right)\\
&\lesssim \beta\beta\|\partial_1^2 v\|_{L^2}\|\partial_1 v\|_{L^2}\\
&\leqslant \beta\frac{1}{2}\nu\|\partial_1\omega\|_{L^2}^2+c\beta^2\|\partial_1 v\|_{L^2}^2,
\end{aligned}
$$

where it used $\int \partial_1 v^{\neq} v^0 \mathrm{d}X=0$ and $\partial_1 v^0=0$ in the third equality. Then inserting the estimate Eq. (11) into Eq. (10), we conclude that:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^2}^2+\nu\|\partial_1\omega\|_{L^2}^2\leqslant c\beta^2\|\partial_1 v\|_{L^2}^2. \tag{12}
$$

Next, we focus on the H^2 estimate of v. By forming the L^2 product of the Eq. $(9)_1$ with $-\Delta\omega$, and utilizing integration by parts, we deduce that:

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega\|_{L^2}^2+\nu\|\partial_1\nabla\omega\|_{L^2}^2&=-\int\nabla\left(v\cdot\nabla\omega\right)\cdot\nabla\omega\mathrm{d}X+\beta\int v_2\,\Delta\omega\mathrm{d}X\\
&:=\mathcal{I}_1+\mathcal{I}_2.
\end{aligned} \tag{13}
$$

We now control the terms on the right-hand side of Eq. (13). Focusing on the first term $\mathcal{I}_1$, we further separate it into two distinct terms:

$$
\begin{aligned}
\mathcal{I}_1&=-\int\nabla\left(v\cdot\nabla\right)\omega\cdot\nabla\omega\mathrm{d}X\\
&=-\int\partial_1\left(v\cdot\nabla\right)\omega\partial_1\omega\mathrm{d}X-\int\partial_2\left(v\cdot\nabla\right)\omega\partial_2\omega\mathrm{d}X\\
&:=\mathcal{I}_{1,1}+\mathcal{I}_{1,2}.
\end{aligned} \tag{14}
$$

For $\mathcal{I}_{1,1}$, using hölder's inequality and embedding inequality $H^1(\mathbb{R}^2)\hookrightarrow L^q(\mathbb{R}^2)(2\leqslant q<\infty)$, we have:

$$
\begin{aligned}
\mathcal{I}_{1,1}&\leqslant\|\partial_1 v\|_{L^6}\|\partial_1\omega\|_{L^3}\|\nabla\omega\|_{L^2}\\
&\leqslant\|\partial_1 v\|_{H^1}\|\partial_1\omega\|_{H^1}\|v\|_{H^2}\\
&\leqslant\|\partial_1 v\|_{H^2}^2\|v\|_{H^2}.
\end{aligned} \tag{15}
$$

For the term $\mathcal{I}_{1,2}$, we also break it into two parts:

$$
\begin{aligned}
\mathcal{I}_{1,2}&=-\int\partial_2\,v_1\partial_1\omega\partial_2\omega\mathrm{d}X-\int\partial_2\,v_2\partial_2\omega\partial_2\omega\mathrm{d}X\\
&:=\mathcal{I}_{1,2,1}+\mathcal{I}_{1,2,2}.
\end{aligned} \tag{16}
$$

We now turn to estimate the term $\mathcal{I}_{1,2,1}$, by the decomposition $v=v^0+v^{\neq}$, $\partial_1\omega=\partial_1\omega^{\neq}$, Lemma 2.1, Eq. (6) and the observation that:

$$
\int\partial_1\,\omega^{\neq}\partial_2\omega^0\partial_2 v_1^0\mathrm{d}X=0,
$$

one has:

$$
\begin{aligned}
\mathcal{I}_{1,2,1}&=-\int\left(\partial_2 v_1^{\neq}+\partial_2 v_1^0\right)\partial_1\omega(\partial_2\omega^{\neq}+\partial_2\omega^0)\mathrm{d}X\\
&=-\int\left(\partial_2 v_1^{\neq}+\partial_2 v_1^0\right)\partial_1\omega^{\neq}(\partial_2\omega^{\neq}+\partial_2\omega^0)\mathrm{d}X\\
&=-\int\left(\partial_2 v_1^0\partial_2\omega^{\neq}\partial_1\omega^{\neq}+\partial_2 v_1^{\neq}\partial_2\omega^0\partial_1\omega^{\neq}+\partial_2 v_1^{\neq}\partial_1\omega^{\neq}\partial_2\omega^{\neq}\right)\mathrm{d}X\\
&\lesssim\|\partial_1\partial_2\omega^{\neq}\|_{L^2}\|\partial_1\partial_2\omega^{\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega^{\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_2 v_1^0\|_{L^2}+\|\partial_1\partial_2 v_1^{\neq}\|_{L^2}\|\partial_1\partial_2\omega^{\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega^{\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_2\omega^0\|_{L^2}\\
&\quad+\|\partial_1\partial_2 v_1^{\neq}\|_{L^2}\|\partial_1\partial_2\omega^{\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega^{\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_2\omega^{\neq}\|_{L^2}\\
&\lesssim\|\partial_1\omega^{\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\omega^{\neq}\|_{L^2}^{\frac{1}{2}}\left(\|\partial_1\partial_2\omega^{\neq}\|_{L^2}\|\partial_2 v_1^0\|_{L^2}+\|\partial_1\partial_2 v_1^{\neq}\|_{L^2}(\|\partial_2\omega^{\neq}\|_{L^2}+\|\partial_2\omega^0\|_{L^2})\right)\\
&\lesssim\|\partial_1\omega\|_{H^1}(\|\partial_1\partial_2\omega\|_{L^2}\|\partial_2 v_1\|_{L^2}+\|\partial_2\omega\|_{L^2}\|\partial_1\partial_2 v_1\|_{L^2})\\
&\lesssim\|\partial_1 v\|_{H^2}^2\|v\|_{H^2}.
\end{aligned}
$$

Similarly, one also gets:

$$
\begin{aligned}
\mathcal{I}_{1,2,2}&=\int\partial_1\,v_1\partial_2\omega\partial_2\omega\mathrm{d}X\\
&=\int\partial_1\,v_1^{\neq}(\partial_2\omega^{\neq}+\partial_2\omega^0)(\partial_2\omega^{\neq}+\partial_2\omega^0)\mathrm{d}X\\
&=\int\left(2\partial_1 v_1^{\neq}\partial_2\omega^{\neq}\partial_2\omega^0+\partial_2\omega^{\neq}\partial_2\omega^{\neq}\partial_1 v_1^{\neq}\right)\mathrm{d}X\\
&\lesssim\|\partial_1\partial_2\omega^{\neq}\|_{L^2}\|\partial_1\partial_2 v_1^{\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_1 v_1^{\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_2\omega^0\|_{L^2}\\
&\quad+\|\partial_1\partial_2\omega^{\neq}\|_{L^2}\|\partial_1\partial_2 v_1^{\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_1 v_1^{\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_2\omega^{\neq}\|_{L^2}\\
&\lesssim\|\partial_1\partial_2 v_1^{\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_1 v_1^{\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\omega^{\neq}\|_{L^2}(\|\partial_2\omega^0\|_{L^2}+\|\partial_2\omega^{\neq}\|_{L^2})\\
&\lesssim\|\partial_1 v\|_{H^1}\|\partial_1\partial_2\omega\|_{L^2}\|\partial_2\omega\|_{L^2}\\
&\lesssim\|\partial_1 v\|_{H^2}^2\|v\|_{H^2}.
\end{aligned}
$$

Combining the above two estimates with Eq. (16), we have:

$$\mathcal{J}_{1,2} \lesssim \|v\|_{H^2}\|\partial_1 v\|_{H^2}^2,$$

which, together with Eq. (15), implies:

$$\mathcal{J}_1 \lesssim \|v\|_{H^2}\|\partial_1 v\|_{H^2}^2. \tag{17}$$

For the linear term $\mathcal{J}_2$, we decompose it into two terms:

$$\begin{aligned}\mathcal{J}_2 &= \beta\int v_2\,\partial_1^2\omega \mathrm{d}X + \beta\int v_2\,\partial_2^2\omega \mathrm{d}X\\ &:= \mathcal{J}_{2,1} + \mathcal{J}_{2,2}.\end{aligned}$$

By hölder's inequality together with Young's inequality, the integrals $\mathcal{J}_{2,1}$ can be bounded:

$$\begin{aligned}\mathcal{J}_{2,1} &= \beta\int \partial_1^2\,\omega^{\neq}\big(v_2^{\neq} + v_2^0\big)\mathrm{d}X\\ &= \beta\int \partial_1^2\,\omega^{\neq} v_2^{\neq}\mathrm{d}X - \beta\int \partial_1\,\omega^{\neq}\partial_1 v_2^0\mathrm{d}X\\ &= \beta\int \partial_1^2\,\omega^{\neq} v_2^{\neq}\mathrm{d}X \lesssim \beta\|\partial_1^2\omega^{\neq}\|_{L^2}\|\partial_1 v_2^{\neq}\|_{L^2}\\ &\leqslant \frac{\nu}{4}\|\partial_1\nabla\omega\|_{L^2}^2 + c\beta^2\|\partial_1 v\|_{L^2}^2.\end{aligned}$$

Similar to the estimate of $\mathcal{J}_{2,1}$, one also has:

$$\begin{aligned}\mathcal{J}_{2,2} &= \beta\int \partial_1 v_1\,\partial_2\omega \mathrm{d}X = \beta\int \partial_1\, v_1^{\neq}\partial_2\omega^{\neq}\mathrm{d}X\\ &\lesssim \beta\|\partial_1 v_1^{\neq}\|_{L^2}\|\partial_1\partial_2\omega^{\neq}\|_{L^2}\\ &\leqslant \frac{\nu}{4}\|\partial_1\nabla\omega\|_{L^2}^2 + c\beta^2\|\partial_1 v\|_{L^2}^2.\end{aligned}$$

By adding the two previous estimates leads to:

$$\mathcal{J}_2 \leqslant \frac{\nu}{2}\|\partial_1\nabla\omega\|_{L^2}^2 + c\beta^2\|\partial_1 v\|_{L^2}^2. \tag{18}$$

Consequently, putting the estimates Eqs. (17) and (18) into Eq. (13) yields that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega\|_{L^2}^2 + \nu\|\partial_1\nabla\omega\|_{L^2}^2 \leqslant c\beta^2\|\partial_1 v\|_{L^2}^2 + c\|v\|_{H^2}\|\partial_1 v\|_{H^2}^2. \tag{19}$$

Then, we deduce from $\frac{4\beta^2}{c}\times$ Eqs. (8), (12) and (19) that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2c\beta^2}{\nu}\|v\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2\right) + 2c\beta^2\|\partial_1 v\|_{L^2}^2 + \nu\|\partial_1\omega\|_{H^1}^2 \lesssim \|v\|_{H^2}\|\partial_1 v\|_{H^2}^2.$$

By integrating the above inequality with respect to time t after using the fact that $\|\omega\|_{H^1}^2 = \|\nabla u\|_{H^1}^2$ and $\|\partial_1\omega\|_{H^1}^2 = \|\partial_1\nabla u\|_{H^1}^2$ gives the desired estimate Eq. (7).

Next, we are ready to establish Theorem 1.1. The local existence of a solution to Eq. (2) for general smooth initial data in a bounded domain Ω can be established using classical methods, as discussed in [22, 23] and references therein. Furthermore, by employing the uniform estimate Eq. (7), we can demonstrate the existence of global solutions for the Eq. (2).

First, one can identify two positive constants c_1 and c_2 such that:

$$\|v\|_{H^2}^2 + \int_0^T \|\,\partial_1 v\|_{H^2}^2\mathrm{d}t \leqslant c_1\|v_{in}\|_{H^2}^2 + c_2 sup_{0\leqslant t\leqslant T}\|v\|_{H^2}\int_0^T \|\,\partial_1 v\|_{H^2}^2\mathrm{d}t.$$

If we make the ansatz that:

$$\|v\|_{H^2}^2 \leqslant \frac{1}{4c_2^2},$$

Then, we can obtain:

$$\|v\|_{H^2}^2 + \frac{1}{2}\int_0^T \|\,\partial_1 v\|_{H^2}^2\mathrm{d}t \leqslant c_1\|v_{in}\|_{H^2}^2. \tag{20}$$

If we choose initial value sufficiently small such that:

$$\|v_{in}\|_{H^2}^2 \leqslant \frac{1}{8c_1c_2^2}, \tag{21}$$

which together with Eq. (20) gives that:

$$\|v\|_{H^2}^2 \leqslant \frac{1}{8c_2^2}. \tag{22}$$

Then, by using a bootstrapping argument, one can confirm that Eq. (20) is valid for all time as long as Eq. (21) is satisfied, thereby completing the proof of Theorem 1.1.

3. Proof of Theorem 1.2

The purpose of this section is to proof Theorem 1.2, which demonstrates that the H^1 norm of oscillatory part $v^{\neq}$ decays exponentially as time tends to infinity.

By using Lemma 2.1 and the relation:

$$\begin{aligned}(v\cdot\nabla v)^0 &= (v\cdot\nabla v^{\neq})^0+(v\cdot\nabla v^0)^0\\ &=(v\cdot\nabla v^{\neq})^0+(v_2\partial_2 v^0)^0+(v_1\partial_1 v^0)^0\\ &=(v\cdot\nabla v^{\neq})^0+(v_2\partial_2 v^0)^0,\end{aligned}$$

we then take the horizontal average of Eq. (2) to obtain:

$$\begin{cases} v_t^0+(v\cdot\nabla v^{\neq})^0+(v_2\partial_2 v^0)^0+(0,\partial_2 P^0)^T=-\beta y v_\perp^0,\\ \partial_2 v_2^0=0,\end{cases}\tag{23}$$

where $v_\perp:=-(v_2,-v_1)^T$.

Taking the difference of Eqs. (2) and (23), we observe that:

$$\begin{cases} v_t^{\neq}+(v\cdot\nabla v^{\neq})^{\neq}+v_2^{\neq}\partial_2 v^0+\nabla P^{\neq}-\nu\partial_1^2 v^{\neq}=-\beta y v_\perp^{\neq},\\ \mathrm{d}iv v^{\neq}=0.\end{cases}\tag{24}$$

Multiplying (24 by $v^{\neq}$ in L^2, we infer that:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^{\neq}\|_{L^2}^2+\nu\|\partial_1 v^{\neq}\|_{L^2}^2 &= -\int \beta y v_\perp^{\neq}\cdot v^{\neq}\mathrm{d}X-\int v_2^{\neq}\,\partial_2 v^0\cdot v^{\neq}\mathrm{d}X\\ &\quad -\int (v\cdot\nabla v^{\neq})^{\neq}\cdot v^{\neq}\mathrm{d}X\\ &:=\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3.\end{aligned}\tag{25}$$

A direct computation for $\mathcal{J}_1$ shows that:

$$\mathcal{J}_1=-\int \beta y(v_1^{\neq}v_2^{\neq}-v_2^{\neq}v_1^{\neq})\mathrm{d}X=0.$$

By applying Lemma 2.1 and Eq. $(2)_2$, $\mathcal{J}_2$ can be upper-bounded by:

$$\begin{aligned}\mathcal{J}_2 &= -\int (v\cdot\nabla v^{\neq}-(v\cdot\nabla v^{\neq})^0)\cdot v^{\neq}\mathrm{d}X\\ &=\int (v\cdot\nabla v^{\neq})^0\cdot v^{\neq}\mathrm{d}X-\int v\cdot\nabla v^{\neq}\cdot v^{\neq}\mathrm{d}X\\ &=\frac{1}{2}\int |v^{\neq}|^2\mathrm{d}iv v\mathrm{d}X+\int v^{\neq}\cdot(v\cdot\nabla v^{\neq})^0\mathrm{d}X\\ &=0.\end{aligned}$$

By using Eqs. (3), (5) and (6), we have:

$$\begin{aligned}\mathcal{J}_3 &\leqslant \|\partial_1 v_2^{\neq}\|_{L^2}\|\partial_2^2 v^0\|_{L^2}^{\frac{1}{2}}\|\partial_2 v^0\|_{L^2}^{\frac{1}{2}}\|v^{\neq}\|_{L^2}\\ &\lesssim \|\partial_1 v^{\neq}\|_{L^2}^2\|v\|_{H^2}\\ &\leqslant c\varepsilon\|\partial_1 v^{\neq}\|_{L^2}^2.\end{aligned}$$

Putting the above three estimates into Eq. (25) yields:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^{\neq}\|_{L^2}^2+(\nu-c\varepsilon)\|\partial_1 v^{\neq}\|_{L^2}^2=0.\tag{26}$$

We now prove $\|\nabla v^{\neq}\|_{L^2}^2$. Multiplying Eq. (24) by $-\Delta v^{\neq}$ in L^2, we can obtain:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v^{\neq}\|_{L^2}^2+\nu\|\partial_1\nabla v^{\neq}\|_{L^2}^2 &= \int \beta y v_\perp^{\neq}\cdot\Delta v^{\neq}\mathrm{d}X-\int \nabla v^{\neq}\cdot\nabla(v_2^{\neq}\partial_2 v^0)\mathrm{d}X\\ &\quad -\int \nabla v^{\neq}\cdot\nabla(v\cdot\nabla v^{\neq})^{\neq}\mathrm{d}X\\ &:=\mathcal{J}_4+\mathcal{J}_5+\mathcal{J}_6.\end{aligned}\tag{27}$$

By the simple computation, we find that:

$$\begin{aligned}\mathcal{J}_4 &= -\int \beta y\partial_1 v_\perp^{\neq}\cdot\partial_1 v^{\neq}\mathrm{d}X-\int \partial_2(\beta y v_\perp^{\neq})\cdot\partial_2 v^{\neq}\mathrm{d}X\\ &=-\int \beta\,\partial_2 v^{\neq}\cdot v_\perp^{\neq}\mathrm{d}X=2\beta\int v_1^{\neq}\,\partial_1 v_1^{\neq}\mathrm{d}X\\ &=0.\end{aligned}$$

Similar to $\mathcal{J}_2$, we can obtain:

$$\begin{aligned}\mathcal{J}_5 &= -\int \nabla v^{\neq}\cdot\nabla(v\cdot\nabla v^{\neq})^0\mathrm{d}X-\int \nabla v^{\neq}\cdot\nabla(v\cdot\nabla v^{\neq})\mathrm{d}X\\ &=-\int \nabla v^{\neq}\cdot\nabla v_1\partial_1 v^{\neq}\mathrm{d}X-\int \nabla v^{\neq}\cdot\nabla v_2\partial_2 v^{\neq}\mathrm{d}X\\ &:=\mathcal{J}_{5,1}+\mathcal{J}_{5,2}.\end{aligned}$$

Using Eqs. (5) and (6), $\mathcal{J}_{5,1}$ can be bounded by:

$$\begin{aligned}\mathcal{J}_{5,1} &= -\int \partial_1 v^{\neq} \cdot \partial_1 v^{\neq} \partial_1 v_1 \mathrm{d}X - \int \partial_1 v^{\neq} \cdot \partial_2 v^{\neq} \partial_2 v_1 \mathrm{d}X \\ &\lesssim \|\partial_1 v_1\|_{H^1} \|\partial_1\partial_2 v^{\neq}\|_{L^2}^{\frac{1}{2}} \|\partial_1 v^{\neq}\|_{L^2}^{\frac{1}{2}} \|\partial_1^2 v^{\neq}\|_{L^2} + \|\partial_2 v_1\|_{H^1} \|\partial_1\partial_2 v^{\neq}\|_{L^2}^{\frac{3}{2}} \|\partial_1 v^{\neq}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\partial_1 \nabla v^{\neq}\|_{L^2}^2 \|v\|_{H^2} \leqslant c\varepsilon \|\partial_1 \nabla v^{\neq}\|_{L^2}^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{J}_{5,2} &= -\int \partial_2 v^{\neq} \cdot \partial_1 v^{\neq} \partial_1 v_2 \mathrm{d}X - \int \partial_2 v^{\neq} \cdot \partial_2 v^{\neq} \partial_2 v_2 \mathrm{d}X \\ &\lesssim \|\partial_1 v_2\|_{H^1} \|\partial_1^2 v^{\neq}\|_{L^2} \|\partial_1\partial_2 v^{\neq}\|_{L^2} + \|\partial_1\partial_2 v^{\neq}\|_{L^2}^2 \|\partial_1 v_1\|_{H^1} \\ &\lesssim \|\partial_1 \nabla v^{\neq}\|_{L^2}^2 \|v\|_{H^2} \leqslant c\varepsilon \|\partial_1 \nabla v^{\neq}\|_{L^2}^2.\end{aligned}$$

Therefore, $\mathcal{J}_5$ is bounded by:

$$\mathcal{J}_5 \leqslant c\varepsilon \|\partial_1 \nabla v^{\neq}\|_{L^2}^2.$$

In a manner similar to the estimate for $\mathcal{J}_5$, one can also derive:

$$\begin{aligned}\mathcal{J}_6 &= \int \partial_2 v^0 \cdot \partial_1^2 v^{\neq} v_2^{\neq} \mathrm{d}X - \int \partial_2 v^{\neq} \cdot \partial_2 (v_2^{\neq} \partial_2 v^0) \mathrm{d}X \\ &\lesssim \|\partial_2 v^0\|_{H^1} \|\partial_1 v_2^{\neq}\|_{L^2} (\|\partial_1\partial_2 v^{\neq}\|_{L^2} + \|\partial_1^2 v^{\neq}\|_{L^2}) + \|\partial_1\partial_2 v^{\neq}\|_{L^2}^2 \|\partial_0 v^0\|_{H^1} \\ &\lesssim \|\partial_1 \nabla v^{\neq}\|_{L^2}^2 \|v\|_{H^2} \leqslant c\varepsilon \|\partial_1 \nabla v^{\neq}\|_{L^2}^2.\end{aligned}$$

Hence, collecting the above three equalities of $\mathcal{J}_4 - \mathcal{J}_6$, together with Eq. (27), we have:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v^{\neq}\|_{L^2}^2 + (\nu - c\varepsilon)\|\partial_1 \nabla v^{\neq}\|_{L^2}^2 \leqslant 0,$$

which, together with Eq. (26), gives that:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^{\neq}\|_{H^1}^2 + (\nu - c\varepsilon)\|\partial_1 v^{\neq}\|_{H^1}^2 \leqslant 0. \tag{28}$$

Noting that,

$$\|v^{\neq}\|_{H^1}^2 \leqslant c\|\partial_1 v^{\neq}\|_{H^1}^2,$$

then we can derive from Eq. (28) that:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|v^{\neq}\|_{H^1}^2 e^{ct}) + (\nu - c\varepsilon - \frac{1}{2}c^2)\|\partial_1 v^{\neq}\|_{H^1}^2 e^{ct} \leqslant 0.$$

Integrating the above result over $(0, t)$ and choosing ε and c sufficiently small, we derive Eq. (4). This concludes the proof of Theorem 1.2.

Author Contributions

Y. Zhang designed the study, initiated the project, and proposed the experiments. H. Liu and M. Liu carried out the mathematical modeling and validated the theoretical results. All authors contributed to writing the manuscript, discussed the findings, and reviewed and approved the final version.

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Conflict of Interest

The authors declare that they have no potential conflicts of interest.

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Data Availability Statements

The paper is self-contained and no external data are used.

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