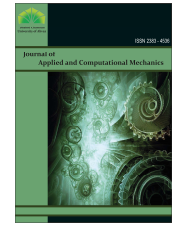




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Research Paper

Gyrostatic Frequency Shifting and Near-Singular Transitions in the Two-Stage Rotational Dynamics of an Impact-Struck Rubble-Pile Asteroid During Hyperbolic Planetary Flyby

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Abstract. The rotational evolution of a rubble-pile asteroid that sustains a near-perpendicular kinetic impact immediately prior to entering the gravitational influence zone of a planet is investigated within the gyrostator framework, wherein the total angular momentum components along the three principal axes are impacted by the constant angular momentum of internally spinning rotors fixed in the carrier body. A complete two-stage semi-analytical treatment is developed: the first stage characterizes the post-impact free rotation under time-varying principal moments of inertia driven by rubble-pile mass ejection, while the second stage analyzes the spin state during hyperbolic planetary approach under gravitational tidal torques. In the first stage, the spin-up law governing the angular velocity's first component is shown to be independent of the gyrostatic bias first component to leading order, and the kinematic relation $\sin\psi/\sin\theta = \text{const}$ is established as an exact first integral of the leading-order reduced Stage-1 kinematics, unaffected by the gyrostatic extension. In the second stage, a second-order ordinary differential equation for the angular momentum second component is derived, whose homogeneous part oscillates at a gyrostatically shifted natural frequency; the correction is obtained in closed form for the squared frequency, $\Delta(\omega^2) = \omega_{gyro}^2 - \omega_0^2$, and is positive throughout the spun-up phase. Numerical integration of the full nonlinear system confirms the analytical predictions and additionally reveals three dynamically significant phenomena absent from the classical formulation: a progressive phase drift between the gyrostatic and classical solutions is accumulated; near-singular transitions at the zero-crossings of the gyrostatic denominators, at which the reduced description becomes singular while the full nonlinear solution remains regular; and a measurable variation of the total angular momentum magnitude produced by the external tidal forcing and modulated by the internal rotor bias. The results establish gyrostatic frequency shifting as a quantitatively predictable and observationally accessible signature of internal rotor angular momentum in small-body spin dynamics, with direct implications for the interpretation of close-flyby light curve data and for planetary-defense mission planning.

Keywords: Asteroid gyrostator rotation, Rubble-pile asteroid, Nonlinear dynamics, Rotational dynamics.

1. Introduction

The rotational evolution of asteroids has held the sustained attention of the celestial-mechanics community, owing to the variety of physical agents capable of reshaping the spin state of these small bodies. Three broad classes of such agents are conventionally distinguished. The first is the gravitational action of an external attracting body on an extended, mass-anisotropic asteroid, which produces gravity-gradient (tidal) torques during close planetary encounters. The second is radiative in origin, the anisotropic thermal re-emission responsible for the YORP effect that slowly reshapes spin rates and obliquities over long time scales [1, 2]. The third consists of impulsive external events: mutual collisions or deliberate projectile impacts, the latter now a demonstrated instrument of planetary defense. The present study is situated at the confluence of the first and third classes and augments them with a structural feature not previously examined jointly with either: a constant internal rotor angular momentum, which renders the carrier asteroid a gyrostator.

The gravitational thread of the problem concerns how tidal torques and spin-orbit coupling govern the orientation of a rotating body during a planetary encounter. Melnikov [3] and Lobanova and Melnikov [4] analyzed the rotational disturbances experienced by asteroids approaching planets, the latter treating the 2029 close approach of (99942) Apophis to the Earth; the same encounter has been examined with constrained spin-state models by Benson et al. [5] and Souchay et al. [6]. On the theoretical side, the interplay of triaxiality and spin-orbit resonance for a rigid body was quantified by Cantero et al. [7], while the chaotic rotation



exemplified by Hyperion [8, 9] established the acute sensitivity of small body spin to gravitational forcing. The structural consequence of such forcing, tidal disruption when internal stresses exceed material strength, was placed on a continuum footing by Holsapple and Michel [10] and modeled for Apophis through finite-element analysis by Hirabayashi et al. [11]. The semi-analytical treatment of asteroid rotation during planetary approach on which the present study builds, was developed by Ershkov [12].

The impulsive thread concerns the rotational response to surface impacts and the attendant mass loss. The cratering and ejection dynamics of a rubble-pile target were modeled by Ershkov and Leshchenko [13] through a dynamic-jamming description [14], and the rotational consequences of an inelastic collision on a non-rigid asteroid were quantified in the same line of work [15, 16], with the trapped-motion analysis of Ershkov et al. [17] supplying the orbital setting. Empirically, the Double Asteroid Redirection Test furnished the first controlled measurement of impact-driven momentum transfer: the kinetic impact on Dimorphos [18] altered the binary orbital period and yielded a momentum-enhancement factor of order three to four [19]. Of direct relevance here, the impact left Dimorphos in an excited, non-principal-axis spin state that evolved quasi-chaotically [20], an observational precedent for impact driven internal spin excitation in a loosely consolidated body. The theoretical spin-orbit response of a binary asteroid to a kinetic impactor was subsequently formalized by Gaitanas et al. [21].

The gyrostatic thread concerns how a constant internal rotor angular momentum reshapes the nonlinear rotational dynamics of a carrier body. In a gyrost, one or more internal rotors with body-fixed angular momenta contribute additive terms to the total angular momentum along each principal axis, introducing additional gyroscopic coupling into the Euler equations and shifting the natural frequency of the precessing spin [22-24]. Several strands of this literature bear on the present problem. Gyrostats carrying internally moving or prescribed-path masses have been studied for their equations of motion and stability [25, 26]; fluid-filled and viscous gyrostats, in which a cavity of viscous liquid produces reaction torques, have been treated by averaging and control-theoretic methods [27-32]. The onset of chaos in gyrost motion, through connections to Lorenz-, Rossler-, and Sprott-type attractors and through Melnikov analysis, has been documented in resistive media [33-35]. Conditions for regular and semi-regular precession of a gyrost in a force field, together with the stability of resonant rotation in an elliptic orbit, were established in [36-38]. Finally, the aerospace applications, attitude dynamics and stabilization of gyrost satellites, dual-spin spacecraft, and damping-based control in resistive media have been developed extensively [39-43].

Although each of these three threads is individually mature, they have been pursued largely in isolation: the tidally forced rigid-body asteroid, the impact-driven rubble pile, and the torque-free gyrost constitute three separate limiting cases. The present work occupies and analyses their common intersection. Relative to the semi-analytical framework of Ershkov [12], which treats a non-gyrostatic asteroid, the contribution here is threefold. First, the carrier body is endowed with a constant internal rotor angular momentum, and the gyrost-augmented Euler system is derived from first principles, reducing to the formulation of [12] in the vanishing-bias limit. Second, a second-order governing equation for the transverse angular-momentum component is obtained whose homogeneous part oscillates at a gyrostatically shifted natural frequency, established here in closed form. Third, the analysis exposes three dynamical signatures absent from the non-gyrostatic case: a progressive phase drift between the gyrostatic and classical solutions, denominator-induced near-singular transitions in the reduced description, and a rotor-modulated variation of the total angular-momentum magnitude under tidal forcing. The study thereby extends, rather than replicates, the foundational analysis, and it draws on the broader corpus above so as not to rest on any single antecedent.

The remainder of the paper is organized as follows. Section 2 assembles the governing system, the gyrost-augmented Euler equations with gravitational torques, the kinematic relations for the Euler angles, and the orbital dynamics, and establishes the physical scale of the internal rotor bias. Section 3 develops the semi-analytical solution for the post-impact kinematic stage. Section 4 analyses the flyby stage, derives the governing second-order equation for the transverse angular momentum, and characterizes the gyrostatically shifted natural frequency. Section 5 presents the numerical results. Section 6 offers a broader discussion, and Section 7 collects the concluding remarks.

2. The Governing System of Equations

Figure 1 presents the schematic configuration of the two-stage problem: an impact-struck rubble-pile asteroid on a hyperbolic approach, together with the internal structure of the gyrost and the orientation of the body frame relative to the space frame. Let the principal moments of inertia of the asteroid satisfy $I_1 \geq I_2 \geq I_3$, and let $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ denote the angular velocity components resolved along the principal axes in the body-fixed frame. In the gyrost formulation, the total angular momentum vector is:

$$\mathbf{K} = (I_1\Omega_1 + \ell_1, I_2\Omega_2 + \ell_2, I_3\Omega_3 + \ell_3), \quad (1)$$

where $\underline{\ell} = (\ell_1, \ell_2, \ell_3)$ are constant projections of the rotor angular momentum onto the principal axes in the body frame. Applying the angular momentum theorem in the rotating frame, $\dot{\mathbf{K}} + \underline{\Omega} \times \mathbf{K} = \mathbf{M}$, and expanding the cross-product component by component, the governing dynamical equations take the form:

$$\begin{aligned} I_1 \frac{d\Omega_1}{dt} + (I_3 - I_2)\Omega_2\Omega_3 + \ell_3\Omega_2 - \ell_2\Omega_3 &= 3 \frac{GM_{\text{planet}}}{r^3} \beta \gamma (I_3 - I_2), \\ I_2 \frac{d\Omega_2}{dt} + (I_1 - I_3)\Omega_1\Omega_3 + \ell_1\Omega_3 - \ell_3\Omega_1 &= 3 \frac{GM_{\text{planet}}}{r^3} \gamma \alpha (I_1 - I_3), \\ I_3 \frac{d\Omega_3}{dt} + (I_2 - I_1)\Omega_1\Omega_2 + \ell_2\Omega_1 - \ell_1\Omega_2 &= 3 \frac{GM_{\text{planet}}}{r^3} \alpha \beta (I_2 - I_1), \end{aligned} \quad (2)$$

where M_{planet} is the planetary mass, G is the gravitational constant, and $r = a(e^2 - 1)/(1 + e \cos f)$ is the asteroid-planet separation along the hyperbolic approach trajectory ($e > 1$), with a the semi-major axis and f the true anomaly [4].

Equations (2) presuppose constant principal moments and are, in that form, the equations of the flyby stage treated in Section 4. During the ejection stage of Section 3, the moments vary with time through the mass-loss law (7), and the angular-momentum theorem must be retained in its unreduced form:

$$\frac{d}{dt} [I_i(t)\Omega_i] + [\underline{\Omega} \times \mathbf{K}]_i = 0, \quad \mathbf{K} = (I_i(t)\Omega_i + \ell_i)_{i=1,2,3}, \quad (2a)$$

or, solved for the angular accelerations:

$$I_i(t) \frac{d\Omega_i}{dt} = -[\underline{\Omega} \times \mathbf{K}]_i - \frac{dI_i(t)}{dt} \Omega_i, \quad (2b)$$



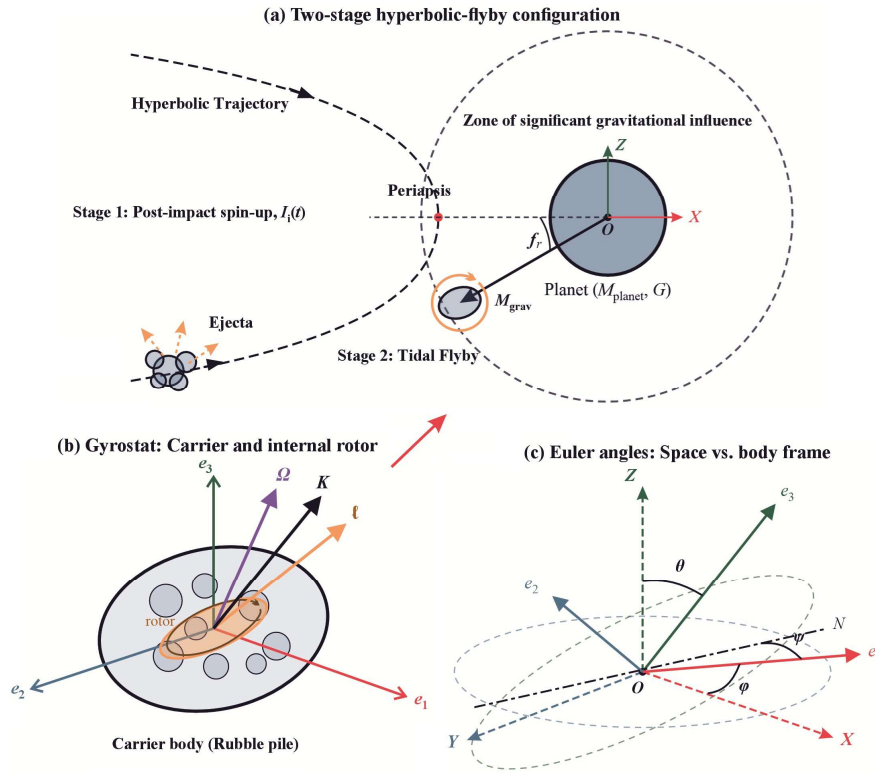


Fig. 1. Schematic configuration of the two-stage problem; (a) Hyperbolic-flyby geometry: Stage 1 (post-impact spin-up with mass ejection and time-varying moments of inertia) occurs outside the zone of significant gravitational influence, while Stage 2 (tidal flyby) unfolds near periaapsis under the gravity-gradient torque, (b) Internal structure of the gyrostat: the rubble-pile carrier, its principal axes, the internal rotor supplying the constant bias, the angular velocity, and the total angular momentum, (c) Orientation of the body frame relative to the space frame through the Euler angles and the line of nodes.

The inertia-rate terms $\dot{I}_i\Omega_i$ are retained in full in the numerical integration of Stage 1 reported in Section 5; Eqs. (2) are used without modification only where the moments are constant. The gravitational torque is absent from Eqs. (2a) and (2b) because the impact is assumed to take place while the asteroid is still outside the zone of significant gravitational influence of the planet [17], so that Stage 1 is torque-free to the accuracy of the present treatment and the tidal torque is switched on only at the onset of Stage 2. It is assumed in addition, following [13, 16], that the ejected material carries away no net angular momentum about the instantaneous center of mass; $\mathbf{K}$ is then conserved throughout the ejection phase and $|\mathbf{K}|$ is an exact invariant of Eq. (2a).

The direction cosines α , β , and γ of the principal axes relative to the planet's direction are given by [3, 4]:

$$\begin{aligned}\alpha &= \cos(\theta - f)\cos\psi - \sin(\theta - f)\sin\psi\sin\phi, \\ \beta &= -\sin(\theta - f)\cos\phi, \\ \gamma &= \cos(\theta - f)\sin\psi + \sin(\theta - f)\cos\psi\sin\phi,\end{aligned}\quad (3)$$

It is noted that the classical (non-gyrostatic) equations studied in [3,12] are recovered from (2) in the limit $\ell \rightarrow 0$. The two new terms in each equation, of the form $\pm\ell_j\Omega_k$, represent the gyroscopic coupling introduced by the internal rotors. Of particular dynamical significance is the term $-\ell_3\Omega_1$ in the second equation, which is of order $\ell_3\Omega_1$ and is therefore not negligible even when $\Omega_2, \Omega_3 \ll \Omega_1$.

The relation between the angular velocity components in the body-fixed frame and the rates of change of the Euler angles $(\dot{\phi}, \dot{\theta}, \dot{\psi})$ is standard [2]:

$$\begin{aligned}\Omega_1 &= \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi, \\ \Omega_2 &= \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi, \\ \Omega_3 &= \dot{\phi}\cos\theta + \dot{\psi},\end{aligned}\quad (4)$$

These relations are purely geometric and are therefore unaffected by the gyrostatic extension. The evolution of the true anomaly with time follows from the conservation of orbital angular momentum along the hyperbolic trajectory [12]:

$$\frac{df}{dt} = \left[\frac{GM_{\text{planet}}}{a^3(e^2 - 1)^3} \right]^{1/2} (1 + e\cos f)^2, \quad (5)$$

For notational economy, the unit of length is henceforth chosen such that $\{GM_{\text{planet}}/[a^3(e^2 - 1)^3]\}^{1/2} = 1$, so that equation (5) reduces to $\dot{f} = (1 + e\cos f)^2$.

The dynamical influence of the internal rotors is governed by the ratio of the rotor angular momentum to the spin angular momentum:

$$\varepsilon_\ell = \frac{\ell}{I\Omega},$$



which is independent of the length normalization adopted above. For the parameter values used in Section 5, this ratio equals 0.04 : the rotors carry about four percent of the body's spin angular momentum. Two considerations establish the physical plausibility of this value. Writing the rotor contribution as $\ell = I_r \omega_{\text{rel}}$ for an interior region of moment I_r turning at a rate ω_{rel} relative to the carrier gives $\varepsilon_\ell = (I_r / I_A)(\omega_{\text{rel}} / \Omega)$. For a coherent interior core of radius fraction λ at the bulk density, this reduces to:

$$\varepsilon_\ell = \lambda^5 \frac{\omega_{\text{rel}}}{\Omega},$$

So that a bias of the required magnitude is furnished by a large interior region ($\lambda \gtrsim 0.7$) in modest differential rotation, of order ten to twenty percent of the spin rate. Such coherent differential rotation is a natural state for a loosely consolidated rubble pile, whereas the contributions of many independently oriented sub-masses cancel as a random walk and cannot build a systematic bias. Alternatively, the bias may be excited by the impact itself: an off-center strike deposits a spin angular momentum $\Delta L \sim \beta m v b_\perp$, and for a target of the Didymos–Dimorphos class, this reaches a substantial fraction of the spin angular momentum so that even partial retention as internal differential motion suffices. The excited, non-principal axis rotation state observed for Dimorphos following the DART impact [20] establishes that a kinetic impact can leave a loosely consolidated body in a complex internal spin state, and in that limited sense it supports the plausibility of impact-driven internal excitation. It does not, however, demonstrate the existence of a persistent internal angular momentum fixed in the body frame, and no such claim is made here. The vector $\underline{\ell}$ should be read as an effective idealization: the lowest-order representation of any internal circulation whose angular momentum is approximately steady in the body frame over the interval of interest, whether that circulation is carried by a differentially rotating core, by a coherent interior flow, or by a discrete rotor.

Two conditions underline this idealization and are assumed throughout. First, $\underline{\ell}$ is taken to be constant in magnitude, which requires that whatever couples the internal circulation to the carrier, granular friction in the interior of a rubble pile being the natural candidate, acts on a timescale long compared with the encounter. Since a hyperbolic close approach is completed in a time of the order of the periapsis passage, whereas frictional erasure of coherent differential rotation in weakly bound aggregate proceeds over many rotation periods, the separation of timescales is comfortable, and the assumption $d\underline{\ell}/dt = 0$ is retained. Second, $\underline{\ell}$ is taken to be fixed in direction relative to the principal axes, which presumes that the axis of the internal circulation is not appreciably reoriented by the tidal torque during the flyby; this is consistent with the smallness of ε_ℓ , since the reorientation torque available to act on the internal circulation is smaller than that acting on the carrier by the same ratio. Relaxing either condition, by allowing $\underline{\ell}(t)$ to evolve under an internal coupling law, is a natural extension and is left to future work.

The spin angular momentum carried by the projectile itself is negligible in this budget and is not represented in $\underline{\ell}$. For a DART class impactor of mass $m_p \approx 5.7 \times 10^2$ kg and characteristic dimension $d \approx 1.3$ m, the moment of inertia about its own center is $I_p \sim m_p d^2 / 6 \approx 1.6 \times 10^2$ kg.m², so that even at a generous tumbling rate of 1 rad.s⁻¹ its intrinsic angular momentum is $L_p \sim 1.6 \times 10^2$ kg.m².s⁻¹. For a Dimorphos-class target of mass $M_A \approx 4.3 \times 10^9$ kg, mean radius $R \approx 80$ m and rotation period 11.9 h, the spin angular momentum is $L_A \sim (2/5)M_A R^2 \Omega \approx 1.6 \times 10^9$ kg.m².s⁻¹. The ratio $L_p / L_A \sim 10^{-7}$ remains below 10^{-6} even when the solar arrays are included in I_p . The impactor, therefore, influences the spin state only through the orbital angular momentum $\Delta L \sim \beta m_p v b_\perp$ delivered by an off-center strike, as described above, and never through its own rotation.

3. Semi-Analytical Solution for the Post-Impact Kinematic Stage

Prior to the impact, long-term inelastic dissipation has driven the asteroid's spin toward preferential rotation about the maximal-inertia axis I_1 [1]. The pre-impact spin state therefore satisfies [17]:

$$\{\Omega_2, \Omega_3\} \ll \Omega_1 \quad \Rightarrow \quad \frac{\Omega_2}{\Omega_1}, \frac{\Omega_3}{\Omega_1} \rightarrow 0 \quad (6)$$

The initial state is accordingly prepared with $|\Omega_{2,3}(0)/\Omega_1(0)| = 10^{-3}$. The leading-order reduction developed below is formally valid while this ratio remains small, and its accuracy degrades monotonically as the ratio grows. As quantified in Section 5, the ratio reaches $\sim 2 \times 10^{-1}$ by the end of the ejection phase and attains order unity within the first few time units of the flyby; the semi-analytical formulae are therefore quantitative only over the earlier portion of each stage, and the full nonlinear system is integrated numerically thereafter. The domain of validity of the analytical description is thus bounded from the outset and is stated explicitly wherever those formulae are compared with the numeric.

The impact of a projectile of mass $M \ll M_A$ (M_A : asteroid mass) perpendicular to the surface ejects rubble-pile material according to the dynamic-jamming model of [13, 17]:

$$m_{\text{plug}} = \frac{\pi \rho}{3} \left[\frac{12 k C_0 M}{\pi \rho} t \right]^{3/4}, \quad (7)$$

where ρ is the surface density of the rubble-pile material, C_0 is a characteristic cratering velocity, and k is a dimensionless standoff distance proportional to the projectile radius. The assumption of no appreciable shape change means that changes in the principal moments of inertia are driven exclusively by the reduction in mass. Conservation of angular momentum $K_1 = I_1 \Omega_1 + \ell_1 = \text{const}$ then implies that as $I_1(t)$ decreases, Ω_1 increases according to:

$$\Omega_1(t) \cong \Omega_1(0) \left(1 - \frac{\pi \rho}{3 M_A} \left[\frac{12 k C_0 M}{\pi \rho} t \right]^{3/4} \right)^{-1}. \quad (8)$$

A first-order Taylor expansion of (8), valid while $m_{\text{plug}} \ll M_A$, gives:

$$\Omega_1(t) \cong \Omega_1(0) \left(1 + \frac{\pi \rho}{3 M_A} \left[\frac{12 k C_0 M}{\pi \rho} t \right]^{3/4} \right). \quad (9)$$

For compactness the ejection parameter is henceforth denoted:

$$\Lambda \cong \frac{\pi \rho}{3 M_A} \left[\frac{12 k C_0 M}{\pi \rho} \right]^{3/4},$$



So that $m_{\text{plug}} / M_A = \Lambda t^{3/4}$ and $I_1(t) = I_1(0)(1 - \Lambda t^{3/4})$; the numerical value quoted in Section 5 refers to this grouping. It is observed that ℓ_1 cancels identically from the angular-momentum balance in both Eqs. (8) and (9): the spin-up law is therefore independent of the gyrostatic bias along the dominant axis, which is a consequence of the fact that $d\ell_1/dt = 0$ and of the leading-order vanishing of the cross-coupling terms $\ell_2\Omega_3 - \ell_3\Omega_2$ under assumption (6).

Substituting assumption (6) into the kinematic equations (4) gives, to leading order:

$$\begin{aligned}\Omega_1(t) &\cong \dot{\phi} \sin \psi \sin \theta + \dot{\theta} \cos \psi, \\ 0 &\cong \dot{\phi} \cos \psi \sin \theta - \dot{\theta} \sin \psi, \\ 0 &\cong \dot{\phi} \cos \theta + \dot{\psi},\end{aligned}\tag{10}$$

From the second and third equations of Eq. (10) one extracts:

$$\dot{\psi} \cong -\dot{\phi} \cos \theta, \quad \dot{\theta} \cong \dot{\phi} \cot \psi \sin \theta,\tag{11}$$

whence the first equation of Eq. (10) yields

$$\dot{\phi} \cong \frac{\Omega_1(t) \sin \psi}{\sin \theta}, \quad \dot{\theta} \cong \Omega_1(t) \cos \psi, \quad \dot{\psi} \cong \Omega_1(t) \sin \psi \cot \theta.\tag{12}$$

Taking the ratio $\dot{\psi}/\dot{\theta}$ from Eq. (12) and separating variables:

$$\cot \psi d\psi \cong -\cot \theta d\theta \quad \Rightarrow \quad \int_{\psi(0)}^{\psi} \frac{d(\sin \psi)}{\sin \psi} \cong - \int_{\theta(0)}^{\theta} \frac{d(\sin \theta)}{\sin \theta},\tag{13}$$

which integrates to yield the conservation law:

$$\sin \psi \sin \theta = \sin \psi(0) \sin \theta(0) = \text{const}.\tag{14}$$

This first integral holds for all $\Omega_1(t)$, confirming that the gyrostatic bias does not alter the kinematic conservation law (14), as it is derived from equations (4) alone.

From the second equation of Eq. (12) together with the first integral (14):

$$\dot{\theta} \cong \Omega_1(t) \sqrt{1 - \left(\frac{\sin \psi(0) \sin \theta(0)}{\sin \theta} \right)^2}.\tag{15}$$

Setting $u = \cos \theta$, $A^2 = 1 - \sin^2 \psi(0) \sin^2 \theta(0)$, and integrating:

$$\arcsin \left(\frac{\cos \theta}{\sqrt{1 - \sin^2 \psi(0) \sin^2 \theta(0)}} \right) \cong C_1 - \Omega_1(0)t - \Omega_1(0) \frac{\pi \rho}{3M_A} \left[\frac{12kC_0}{\pi \rho} M \right]^{3/4} t^{7/4},\tag{16}$$

where C_1 is a constant of integration determined by the initial condition. Inverting Eq. (16) gives the explicit time evolution of the nutation angle:

$$\theta(t) \cong \arccos \left[\sqrt{1 - \sin^2 \psi(0) \sin^2 \theta(0)} \sin \left[C_1 - \Omega_1(0)t - \Omega_1(0) \frac{\pi \rho}{3M_A} \left(\frac{12kC_0}{\pi \rho} M \right)^{3/4} t^{7/4} \right] \right],\tag{17}$$

subject to the validity condition $\sin \theta > \sin \psi(0) \sin \theta(0)$. The corresponding evolution of the spin angle follows from Eq. (14):

$$\sin \psi(t) \cong \frac{\sin \psi(0) \sin \theta(0)}{\sin \theta(t)},\tag{18}$$

and the precession rate is given by:

$$\dot{\phi} \cong \frac{\Omega_1(t) \sin \psi(0) \sin \theta(0)}{\sin^2 \theta(t)},\tag{19}$$

which is in general resolvable only through numerical quadrature. The angle ψ characterizes the inclination of the rotation axis to the orbital plane [4].

The analytical reduction $\psi \equiv 0$, $\phi \equiv 0$ adopted below corresponds to the strict limit $\Omega_2, \Omega_3 \rightarrow 0$. In the numerical integration, the transverse components are small but nonzero, so the precession rate $\dot{\phi} = (\Omega_1 \sin \psi + \Omega_2 \cos \psi)/\sin \theta$ does not vanish identically. As shown in Section 5, the rapid excursions of $\phi(t)$ and $\psi(t)$ near the midpoint of the ejection phase coincide with the passage of the nutation angle through $\theta \approx \pi$, at which the 3-1-3 Euler parametrization is degenerate; these excursions are a property of the angular coordinates rather than of the motion, and the direction cosine γ underlying the Stage 2 reduction evolves continuously through the event.

4. Stage 2 Dynamics: Euler Equations Under Gravitational Torques

Selecting the initial condition $\psi(0) = 0$ implies, via the first integral (14), that $\psi \equiv 0$ throughout Stage 1 and hence also at the start of Stage 2. From Eq. (19), it follows that $\dot{\phi} = 0$, so $\phi \equiv \phi(0) = 0$ as well. With $\psi = \phi = 0$, the direction cosine γ in Eq. (3) vanishes identically, which eliminates the gravitational torque from the first Euler equation in Eq. (2). Equation (17) then simplifies to:

$$\theta(t) \cong \arccos \left[\sin \left[C_1 - \Omega_1(0)t - \frac{4}{7} \Omega_1(0) \frac{\pi \rho}{3M_A} \left(\frac{12kC_0}{\pi \rho} M \right)^{3/4} t^{7/4} \right] \right].\tag{20}$$



The gyrostat equations (2) may be written compactly as:

$$\frac{d\mathbf{K}}{dt} + \mathbf{\Omega} \times \mathbf{K} = \mathbf{M}, \quad (21)$$

where $\mathbf{M}$ is the gravitational torque vector. With $\gamma = 0$, the right-hand side of the first scalar equation in Eq. (21) vanishes, while the left-hand side contains the gyrostatic cross-coupling terms $\ell_3\Omega_2 - \ell_2\Omega_3$ of order $O(\ell_i\varepsilon)$ ($\varepsilon = |\Omega_{2,3}|/\Omega_1$), which are negligible under assumption (6). Consequently, to leading order:

$$K_1 = I_1\Omega_1 + \ell_1 \approx \text{const} \equiv K_1^{(0)}. \quad (22)$$

Setting $\gamma = 0$ in the second equation of (2) and writing $\dot{K}_2 = I_1\dot{\Omega}_1$:

$$\dot{K}_2 + [(I_1 - I_3)\Omega_1 + \ell_1]\Omega_3 - \ell_3\Omega_1 = 0. \quad (23)$$

Defining the gyrostatic denominator $\tilde{D}_1 = (I_3 - I_1)\Omega_1 - \ell_1$, Eq. (23) yields:

$$\Omega_3 \cong \frac{\dot{K}_2 - \ell_3\Omega_1}{\tilde{D}_1}, \quad \tilde{D}_1 = (I_3 - I_1)\Omega_1 - \ell_1. \quad (24)$$

The classical result $\Omega_3 = \dot{K}_2/[(I_3 - I_1)\Omega_1]$ is recovered in the limit $\ell \rightarrow 0$, confirming consistency with the original formulation [12]. With $\gamma = 0$, the third equation of (2) reads:

$$\frac{d}{dt}(I_3\Omega_3) + (I_2 - I_1)\Omega_1\Omega_2 + \ell_2\Omega_1 - \ell_1\Omega_2 \cong -\frac{3}{2}(1 + e \cos f)^3 \sin[2(\theta - f)](I_2 - I_1), \quad (25)$$

coupled with $\dot{f} = (1 + e \cos f)^2$. The nonlinear group $(I_2 - I_1)\Omega_1\Omega_2 + \ell_2\Omega_1 - \ell_1\Omega_2$ is simplified by introducing the dimensionless ratios $A = I_2/I_1$, $B = I_3/I_1$, and substituting $\Omega_2 = (K_2 - \ell_2)/I_2$:

$$(I_2 - I_1)\Omega_1\Omega_2 + \ell_2\Omega_1 - \ell_1\Omega_2 = \frac{\tilde{D}_1 K_2 + \ell_2 K_1}{I_2}, \quad (26)$$

where $\tilde{D}_2 = (I_2 - I_1)\Omega_1 - \ell_1$ and the identity $I_2\Omega_1 - \tilde{D}_2 = K_1$ has been used. Differentiating expression (24) for Ω_3 with respect to time yields:

$$I_3 \frac{d\Omega_3}{dt} = \frac{I_3}{\tilde{D}_1} \dot{K}_2 + I_3 \frac{d(1/\tilde{D}_1)}{dt} \dot{K}_2 + \frac{I_3 \ell_1 \ell_3}{\tilde{D}_1^2} \dot{\Omega}_1. \quad (27)$$

Substituting Eqs. (26) and (27) into Eq. (25) and multiplying through by $\tilde{D}_1/I_3$ gives the governing second-order equation for $K_2(t)$:

$$\ddot{K}_2 + \tilde{D}_1 \frac{d(1/\tilde{D}_1)}{dt} \dot{K}_2 + \frac{\tilde{D}_1 \tilde{D}_2}{I_2 I_3} K_2 = \frac{\tilde{D}_1}{I_3} F_{\text{grav}}(f, t) - \frac{\ell_2 K_1 \tilde{D}_1}{I_2 I_3} - \frac{\ell_1 \ell_3}{\tilde{D}_1} \dot{\Omega}_1, \quad (28)$$

with forcing function:

$$F_{\text{grav}}(f, t) = -\frac{3}{2}(1 + e \cos f)^3 \sin[2(\theta(t) - f)](I_2 - I_1), \quad (29)$$

and where $\theta(t)$ is given by Eq. (20); the spin rate $\Omega_1(t)$ of Eq. (9) enters Eq. (28) through the denominators $\tilde{D}_{1,2}$ and through $\dot{\Omega}_1$. The complete Stage 2 system consists of Eq. (28) coupled to the orbital equation $\dot{f} = (1 + e \cos f)^2$.

When Ω_1 is treated as approximately constant, which is justified at the final stage of mass ejection when the plug growth rate diminishes, both $\tilde{D}_1$ and $\tilde{D}_2$ are constant and $\dot{\Omega}_1 = 0$. Equation (28) then reduces to a linear forced oscillator:

$$\ddot{K}_2 + \omega_{\text{gyro}}^2 K_2 = \tilde{F}(f, t), \quad (30)$$

where the gyrostatically shifted natural frequency is:

$$\omega_{\text{gyro}}^2 = \frac{\tilde{D}_1 \tilde{D}_2}{I_2 I_3} = \frac{[(I_3 - I_1)\Omega_1 - \ell_1][(I_2 - I_1)\Omega_1 - \ell_1]}{I_2 I_3}, \quad (31)$$

and the forcing collects the gravitational and gyrostatic constant terms:

$$\tilde{F}(f, t) = \frac{\tilde{D}_1}{I_3} F_{\text{grav}}(f, t) - \frac{\ell_2 K_1 \tilde{D}_1}{I_2 I_3}. \quad (32)$$

Expanding the product in Eq. (31) explicitly identifies the gyrostatic correction to the classical natural frequency:

$$\omega_0^2 = (I_3 - I_1)(I_2 - I_1)\Omega_1^2 / (I_2 I_3) \quad (33)$$

$$\omega_{\text{gyro}}^2 = \omega_0^2 + \frac{\ell_1 \Omega_1 (2I_1 - I_2 - I_3) + \ell_1^2}{I_2 I_3}. \quad (34)$$

For a body with $I_1 > I_2 \geq I_3$, the coefficient $2I_1 - I_2 - I_3$ is positive, so the correction $\ell_1[(2I_1 - I_2 - I_3)\Omega_1 + \ell_1]/(I_2 I_3)$ is positive for every $\Omega_1 > \ell_1/(2I_1 - I_2 - I_3)$, a bound satisfied throughout the spun-up phase: the gyrostatic bias stiffens the oscillatory response



whenever the body spins in the sense of its relaxed state. A change of sign of the correction requires Ω_1 to fall below the small negative value $-\ell_1 / (2I_1 - I_2 - I_3)$, a situation that arises transiently in Stage 2 when the tidal torque drives Ω_1 through zero, and which underlies the frequency softening reported in Section 5.

Two distinct measures of the gyrostatic correction must be kept apart, and the distinction is maintained throughout the remainder of the paper. The analysis delivers directly a correction to the squared frequency:

$$\Delta(\omega^2) = \omega_{\text{gyro}}^2 - \omega_0^2 = \frac{\ell_1 [\Omega_1 (2I_1 - I_2 - I_3) + \ell_1]}{I_2 I_3}, \quad (35)$$

which is a smooth function of Ω_1 , is defined for every Ω_1 irrespective of the sign of ω_{gyro}^2 , and is the quantity reported in the figures. Where $\omega_{\text{gyro}}^2 > 0$, the associated shift of the frequency itself follows from Eq. (35) as:

$$\Delta\omega = \omega_{\text{gyro}} - \omega_0 = \frac{\Delta(\omega^2)}{\omega_{\text{gyro}} + \omega_0}. \quad (36)$$

where $\omega_{\text{gyro}}^2 < 0$, by contrast, the reduced oscillator is locally hyperbolic rather than oscillatory, ω_{gyro} is not real, and $\Delta\omega$ is undefined; only $\Delta(\omega^2)$ retains meaning there. Reporting $\Delta(\omega^2)$ rather than $\Delta\omega$, therefore avoids the spurious discontinuity that a modulus-based definition of the frequency would introduce at the zero-crossings of $\bar{D}_{1,2}$.

5. Numerical Results

The full nonlinear system, equations (2a)–(2b) together with Eq. (4) in Stage 1, where the moments $I_i(t)$ vary and the inertia-rate terms $\dot{I}_i \Omega_i$ are retained in full, and Eqs. (2), (4), and (5) with constant post-ejection moments in Stage 2, was integrated numerically using a variable-step Runge-Kutta (4,5) scheme with absolute and relative tolerances of 10^{-13} and 10^{-11} , respectively, implemented in MATLAB R2026a; the numerical solution is compared throughout against the leading-order semi-analytical predictions. The parameter values adopted throughout are $I_1 = 1.00, I_2 = 0.80, I_3 = 0.60$ (all normalized), gyrostatic bias $\underline{\ell} = (0.020, 0.012, 0.008)$, initial spin $\Omega_1(0) = 0.50$ with $\Omega_{2,3}(0) = 5 \times 10^{-4}$, mass-ejection parameter $\Lambda = 0.018$ (in normalized time units of exponent 3/4), impact duration $t_{\text{impact}} = 8.0$, orbital eccentricity $e = 1.10$, and initial true anomaly $f_0 = -1.80\text{rad}$.

Figure 2 collects the post-impact dynamics during the mass-ejection phase. Panel (a) confirms the anticipated spin-up of $\Omega_1(t)$, with the numerical solution tracking the exact angular-momentum balance (8) closely while diverging from it by a small but steadily growing margin over $t \in [0, 8]$. This divergence is attributable to the gyrostatic cross-coupling terms $\ell_2 \Omega_3 - \ell_3 \Omega_2$ entering the first Euler equation, which are absent from the purely analytical formula (9). The fact that the numerical Ω_1 falls slightly below the analytical prediction indicates that the gyrostatic coupling partially opposes the spin-up, in physical terms corresponding to a transfer of angular momentum from the dominant axis to the transverse degrees of freedom. Panel (b) reveals the evolution of the minor angular velocity components Ω_2 and Ω_3 , which, although initialized at the level $10^{-3} \Omega_1$ demanded by assumption (6), undergo a systematic and asymmetric departure from their initial values. This departure is driven by the gyrostatic forcing terms $-\ell_3 \Omega_1$ and $\ell_2 \Omega_1$ in the second and third Euler equations, respectively. Since these terms are of order $\ell_i \Omega_1$, they are not negligibly small even when $\Omega_{2,3}/\Omega_1 \ll 1$, and they inject a secular bias into the transverse spin components. The opposing signs of the responses reflect the sign structure of the gyrostatic vector $\underline{\ell}$, and this asymmetry would be entirely absent in the classical formulation. Panel (c) displays the nutation angle $\theta(t)$, which rises monotonically to $\theta \approx \pi$ near $t \approx 4.5$ and decreases thereafter. Panel (g) monitors assumption (6) directly: the ratios $|\Omega_{2,3}/\Omega_1|$, initialized at 10^{-3} , cross 10^{-2} by $t \approx 0.6$ and reach $|\Omega_2/\Omega_1| \approx 2 \times 10^{-1}$ at the end of the ejection phase. The leading-order expressions (9), (17) and (20) are accordingly quantitative over the earlier part of Stage 1 and progressively less so towards its end, which accounts for the growing separation visible in panel (a); the curves compared against them are in every case the full nonlinear solution of Eqs. (2a) and (4). Panels (d) and (e) show that the precession angle $\phi(t)$ and spin angle $\psi(t)$ both undergo rapid sigmoidal transitions near the midpoint of the ejection phase. As established in Section 3, these transitions coincide with the passage of the nutation angle through $\theta \approx \pi$, where the 3-1-3 Euler chart is degenerate (gimbal lock); they are a coordinate artifact rather than a physical reorientation. A quaternion re-integration of Stage 1 (Fig. 11) confirms that the orientation and the body axes expressed in the inertial frame evolve smoothly through the event while the quaternion remains a unit norm to machine precision. Panel (f) quantifies the departure from exact K_1 conservation. The three-dimensional angular velocity trajectory in panel (h) traces a smooth, monotonically evolving curve, and panel (i) confirms the proportional decay of all three principal moments of inertia under the mass-ejection model (7).

The onset of Stage 2, governed by the full system (2) with constant post-ejection moments, produces a qualitative change in the character of the rotational dynamics, as seen in Fig. 3. The dominant spin component Ω_1 , nearly monotonic during Stage 1, now undergoes large-amplitude oscillations about its initial value in panel (a), driven by the gravitational tidal torques whose coefficient $3(1 + e \cos f)^3$ peaks sharply at periapsis. The tidal torque magnitude, shown in panel (i), exhibits a corresponding impulse at periapsis near $t \approx 1$, followed by a rapid power-law decay consistent with the r^{-3} dependence as the asteroid recedes along the hyperbolic trajectory; by $t \approx 3$ the torque has fallen to a negligible level. The remainder of the integration window is therefore effectively torque-free, and the continuing oscillations of panel (a) are free gyrostatic precession rather than driven motion, a distinction that should be kept in view when reading the later part of the record. This impulsive character of the gravitational interaction is the rotational analogue of the tidal disruption discussed in [10, 11] for structural integrity. Panel (b) collects the Euler angles: the nutation angle $\theta(t)$ passes from the slow variation inherited from Stage 1 to rapid oscillations with a modulated envelope, while the precession angle $\phi(t)$ exhibits approximately linear secular growth with superimposed oscillatory modulation. Panel (c) shows the true anomaly $f(t)$ together with the orbital separation $r(t)$ along the hyperbolic trajectory. Panel (d) shows that all three total angular momentum components K_i develop oscillations of comparable amplitude during Stage 2. The magnitude $|\underline{K}|$ in panel (e) rises from 0.50 to 1.50 during the periapsis passage and is constant thereafter, consistently with the decay of the torque documented in panel (i). It should be emphasized that the normalization adopted in Section 2, in which $\{GM_{\text{planet}}/[a^3(e^2 - 1)^{3/2}]\}$ is set to unity, places the tidal torque and the spin angular momentum on comparable scales. The resulting threefold change $|\underline{K}|$ is far larger than that produced by a physical planetary encounter, in which the fractional change is minute; the parameter set adopted here should therefore be read as a numerical stress test of the gyrostatic formulation, chosen to render the gyrostatic effects visible within a single figure, rather than as a model of a specific flyby. Its evolution is governed exactly by $d|\underline{K}|^2/dt = 2\underline{K} \cdot \underline{M}$, since the internal term satisfies $\underline{K} \cdot (\underline{\Omega} \times \underline{K}) \equiv 0$; the magnitude; therefore, it responds solely to the external gravitational torque, which for the adopted eccentricity $e = 1.10$ does not vanish over the integration window. The gyrostatic bias enters only through its influence on the direction cosines and on the component exchange among the K_i ; it modulates the



amplitude of the tidal-driven change without generating any drift of its own. A controlled four-case comparison (Fig. 8) confirms this: with the tidal torque suppressed, $|\mathbf{K}|$ is conserved to machine precision for both the gyrostatic and the classical body, whereas the two tidally forced cases exhibit near-identical growth that differs by only a few percent. The comparison of ω_{gyro} with ω_0 in panel (f) confirms the analytical prediction of Eq. (34): during intervals when $|\Omega_1|$ is large, ω_{gyro} exceeds ω_0 , while near the zeros of Ω_1 the correction changes sign, leading to brief periods of frequency softening. The phase portrait of (Ω_2, Ω_3) in panel (g) traces an approximately elliptical closed orbit that evolves secularly in size and orientation. The rotational kinetic energy in panel (h) undergoes a sharp jump at periastris and subsequently reaches a quasi-stationary plateau.

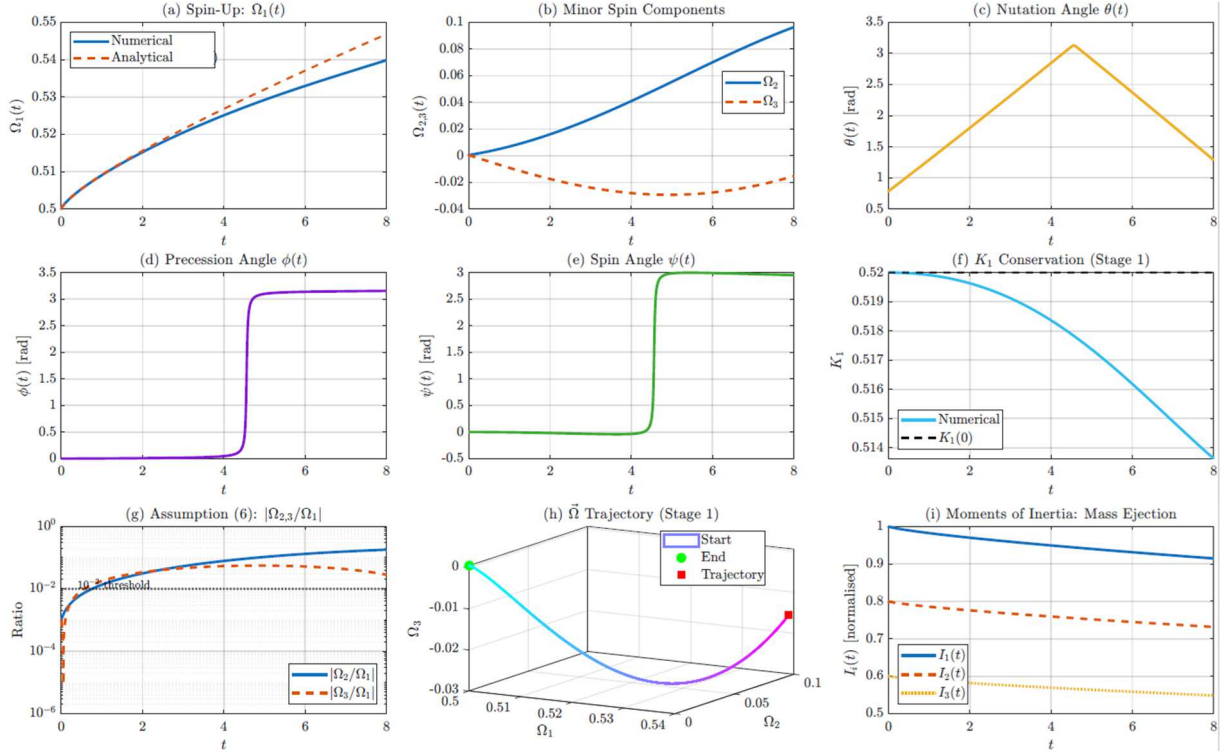


Fig. 2. Stage 1: post-impact dynamics during the mass-ejection phase, obtained from the full nonlinear integration of Eq. (2a) with the kinematics (4).

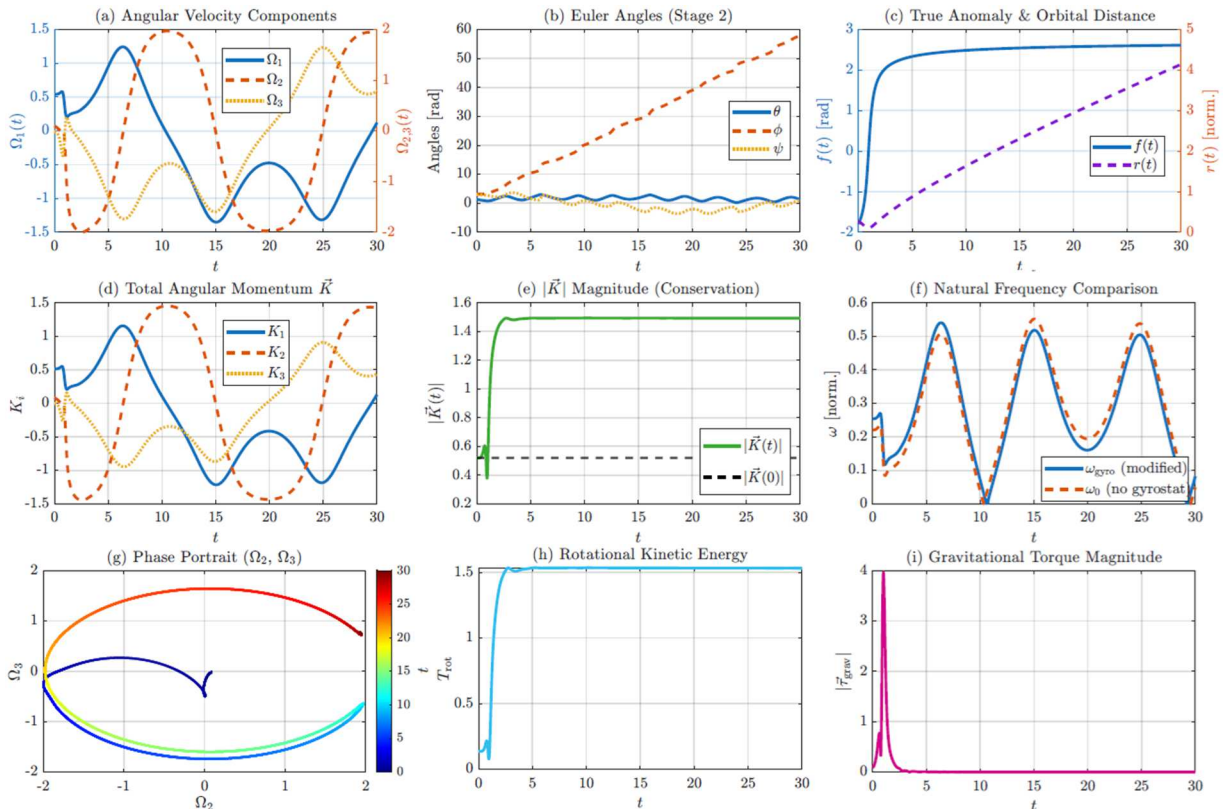


Fig. 3. Stage 2: gyrostat asteroid during close planetary approach, from the full nonlinear system (2), (4), and (5) with constant post-ejection moments.



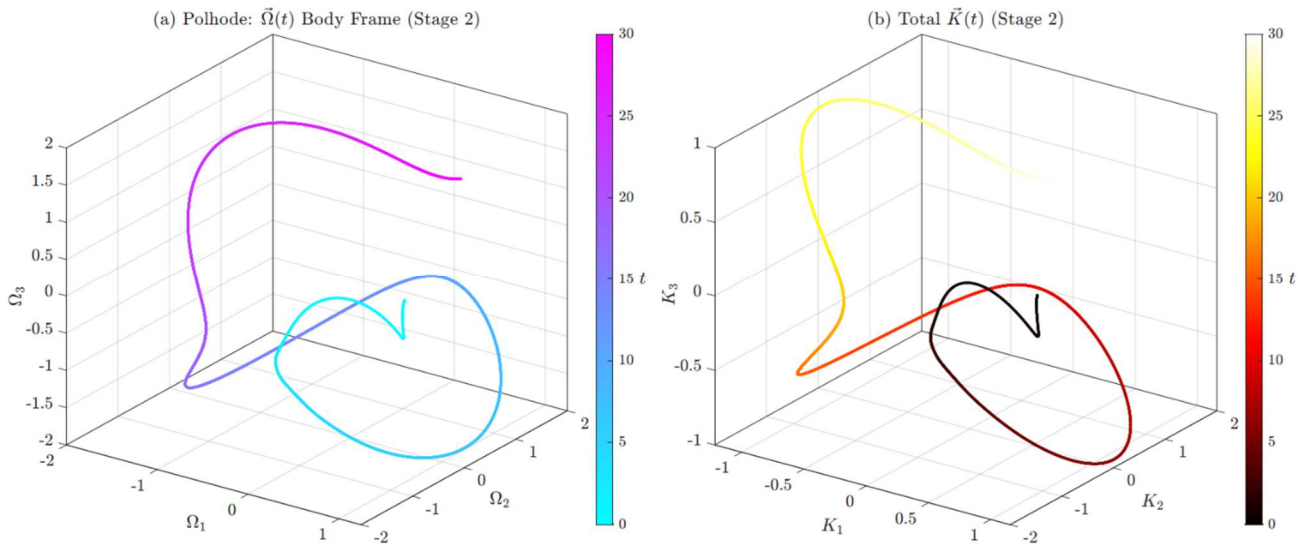


Fig. 4. Geometric view of the Stage-2 dynamics in the body frame, colored by time.

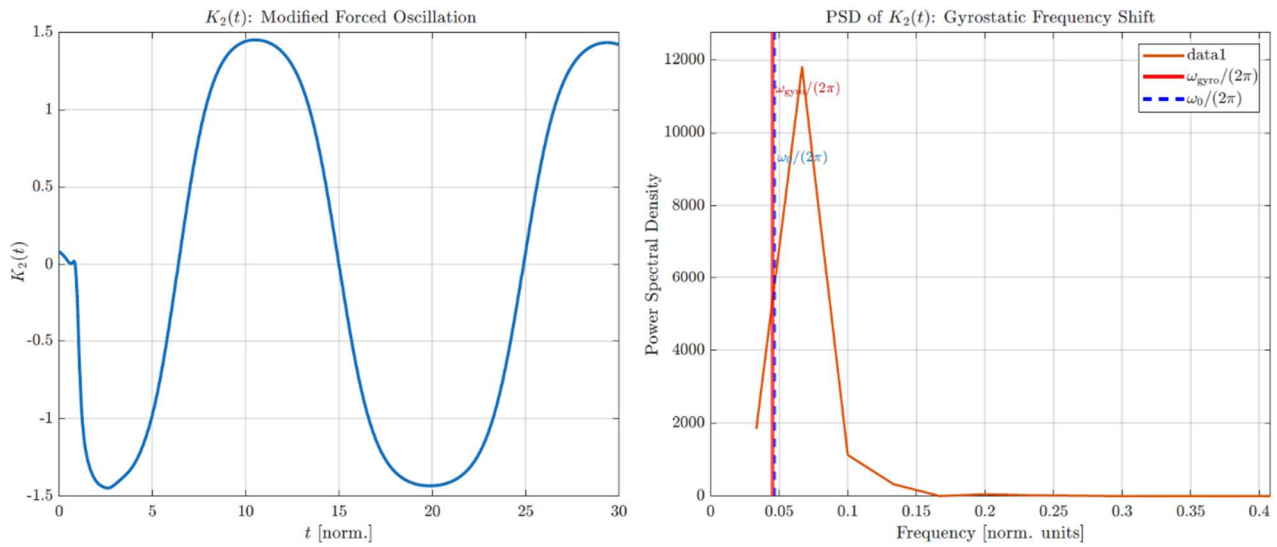


Fig. 5. Forced oscillation of $K_2(t)$ during Stage 2, providing a qualitative test of Eq. (28).

Figure 4 provides a geometric perspective on the Stage 2 dynamics through the three-dimensional trajectories of $\vec{\Omega}(t)$ (the polhode) and $\vec{K}(t)$ in the body frame. The polhode in panel (a) exhibits a structurally complex open trajectory that departs markedly from the closed elliptic curves of the torque-free rigid body. The angular momentum vector $\vec{K}(t)$ in panel (b) traces a similarly open orbit in K -space, with the deviation from a fixed point providing a visualization of the tidally driven variation of $|\vec{K}|}$ already quantified in Fig. 3(e).

Figure 5 presents the time series of $K_2(t)$ and its power spectral density, providing a direct numerical test of Eq. (28). Following the initial transient, $K_2(t)$ settles into a nearly sinusoidal oscillation, consistent with the interpretation of Eq. (30) as a linearly forced oscillator in the quasi-steady regime. The offset of the oscillation center from zero by the amount ℓ_2 is a direct consequence of the gyrostatic bias entering the definition $K_2 = I_2\Omega_2 + \ell_2$. Because $\Omega_1(t)$ oscillates through zero during Stage 2, the shifted frequency ω_{gyro} is itself time-varying, and the spectral peak of a Stage-2 window is intrinsically broadened; over $T = 30$ the record contains barely two oscillation cycles, and the discrete Fourier bin width $\Delta f = 1/T \approx 3.3 \times 10^{-2}$ exceeds the predicted separation of the two theoretical markers by nearly an order of magnitude. The dominant bin consequently falls near, but not on, either marker, and no inference whatever about ω_{gyro} should be drawn from it. The panel is retained solely to exhibit the offset of the oscillation center by ℓ_2 ; the quantitative, resolution-controlled validation of ω_{gyro} is carried out separately in Fig. 9, where the integration window is long enough for the bin width to fall far below the shift.

Figure 6 presents a direct comparison between the gyrostatic ($\ell \neq 0$) and classical ($\ell = 0$) solutions over the full duration of Stage 2. Over the first two-or-three-time units the two solutions are practically indistinguishable, and the difference accumulates progressively thereafter. The divergence is not confined to phase. In panel (a) the classical Ω_1 remains positive throughout the window, whereas the gyrostatic Ω_1 is driven below zero after $t \approx 12$; the envelopes of $\theta(t)$ in panel (d) likewise differ visibly over the second half of the record. The accumulating phase difference is nevertheless the most robust single indicator of the gyrostatic perturbation, and it is quantified in Fig. 10(f). The precession angle $\phi(t)$ in panel (e) exhibits nearly identical secular growth rates in both cases. The phase portrait in panel (f) shows that the two cases trace orbits of comparable radius in the (Ω_2, Ω_3) plane; the gyrostatic bias displaces the center of the small-oscillation orbit to the relative equilibrium $\dot{\Omega}_2 = -\ell_2\Omega_1 / \bar{D}_2$, $\dot{\Omega}_3 = -\ell_3\Omega_1 / \bar{D}_1$ of the torque-free problem, an offset that is small compared with the orbit radius attained at this forcing amplitude and is therefore not resolved visually in this panel.



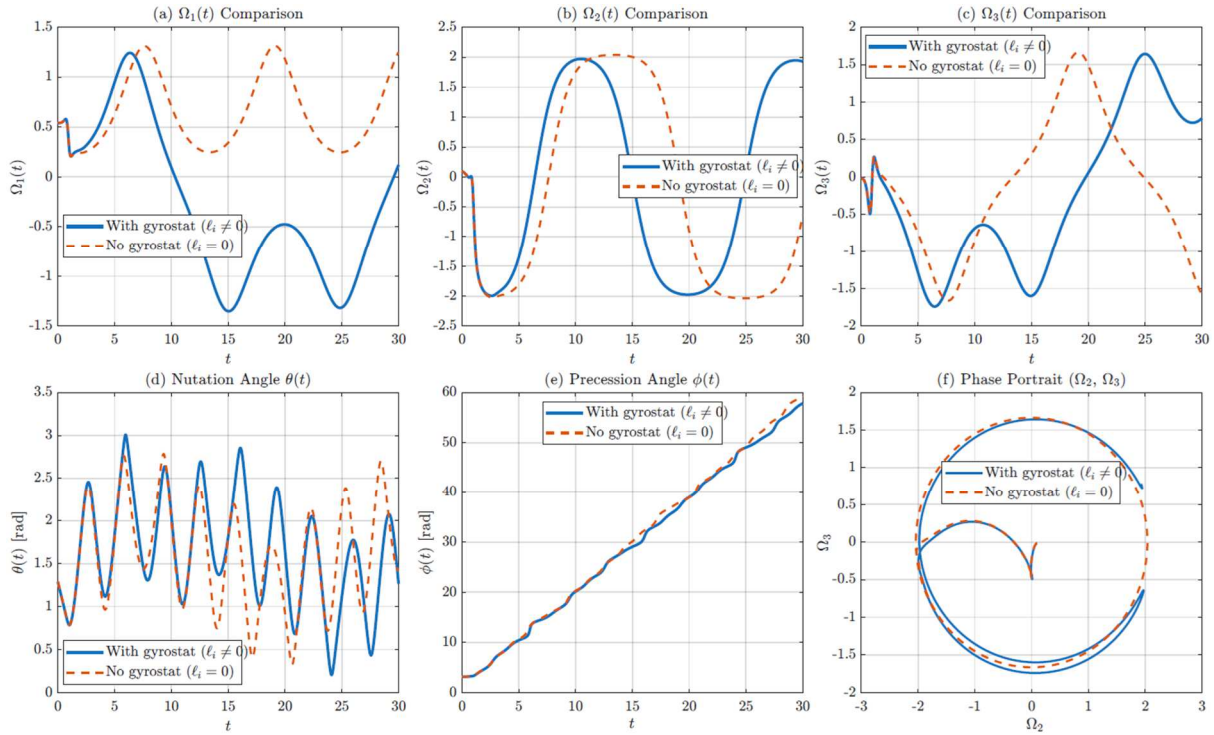


Fig. 6. Direct comparison of the gyrostatic ($\ell \neq 0$) and classical ($\ell = 0$) solutions over the whole of Stage 2, from identical initial conditions.

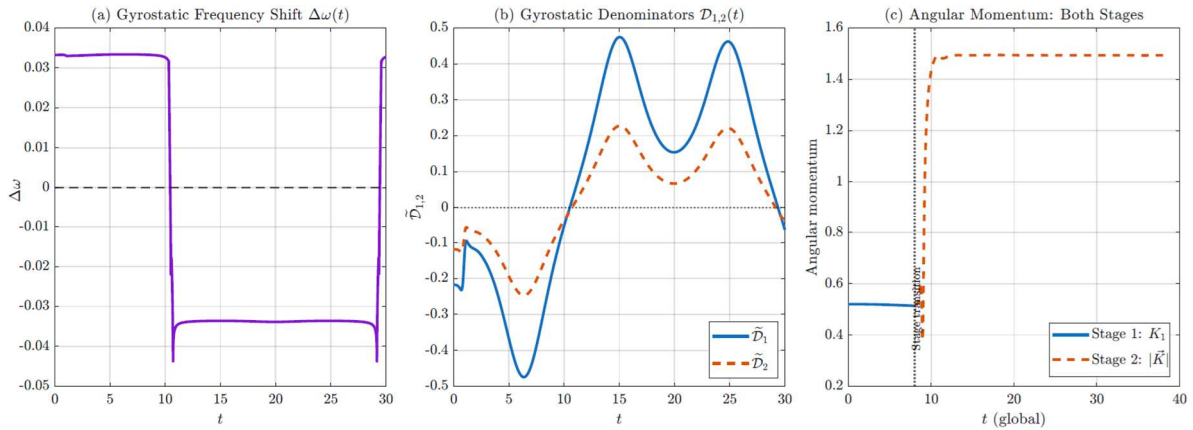


Fig. 7. The gyrostatic correction across the complete two-stage mission.

Figure 7 synthesizes the gyrostatic correction across the complete two-stage mission. Panel (a) reports the correction to the squared frequency, $\Delta(\omega^2)$ of Eq. (35). This is the quantity supplied directly by the analysis, and it remains well-defined and continuous across the intervals in which $\omega_{\text{gyro}}^2 < 0$, where $\Delta\omega$ itself would not exist. It is positive during the early part of Stage 2, where Ω_1 is large and relatively steady, reverses sign when the tidal forcing drives Ω_1 below the threshold $-\ell_1/(2I_1 - I_2 - I_3)$ identified in Eq. (34), and recovers in the final phase of the record. Panel (b) shows the time evolution of the two gyrostatic denominators $\tilde{\mathcal{D}}_1(t)$ and $\tilde{\mathcal{D}}_2(t)$, which change sign twice over the simulation window. At these zero crossings the algebraic reduction leading to Eq. (24) becomes singular and ill-conditioned in their immediate neighborhood, whereas the full nonlinear system (2) remains regular: the diagnostics of Figs. 10(d), 10(e) show that the true Ω_3 stays bounded throughout and that the adaptive step size of the integrator does not collapse. The reduced relation $\Omega_3 = (\tilde{K}_2 - \ell_3\Omega_1)/\tilde{\mathcal{D}}_1$ simply loses validity there, and no conclusion should be drawn from it regarding the magnitude of the physical derivatives. Because ω_{gyro}^2 changes sign across the crossing, the reduced oscillator passes from locally oscillatory to locally hyperbolic behavior, which is reminiscent of a separatrix passage in the torque-free gyrostat [22]; the resemblance is offered as a qualitative analogy only, no separatrix of the present system being crossed. Panel (c) presents $|\tilde{K}|$ across both stages on a unified time axis, the Stage-1 branch being an exact invariant of Eq. (2a) and the Stage-2 branch responding to the external torque alone.

To disentangle the source of the variation in $|\tilde{K}|$, Fig. 8 reports a controlled comparison of the four cases obtained by switching the tidal torque and the gyrostatic bias on and off. With the torque suppressed, the magnitude is conserved to machine precision for both the gyrostatic and the classical body, while the two tidally forced cases drift by nearly identical amounts. This establishes that the change in $|\tilde{K}|$ is produced by the external torque and merely modulated by the internal rotor bias, consistent with the exact identity $d|\tilde{K}|^2/dt = 2\tilde{K} \cdot M$.

Figure 9 validates the gyrostatically shifted natural frequency in its regime of derivation, the torque-free major-axis spin with steady Ω_1 , where the integration window can be made long enough that the spectral bin width falls far below the shift. The measured dominant frequency of K_2 tracks $\omega_{\text{gyro}}/2\pi$ across sweeps in the bias ℓ_1 and in the inertia ratio I_3/I_1 , while departing from the classical $\omega_0/2\pi$ by up to two orders of magnitude more; a Hilbert instantaneous-frequency estimate confirms the stationarity of the oscillation. The error curves in panel (e) quantify the agreement against the reported bin width Δf .



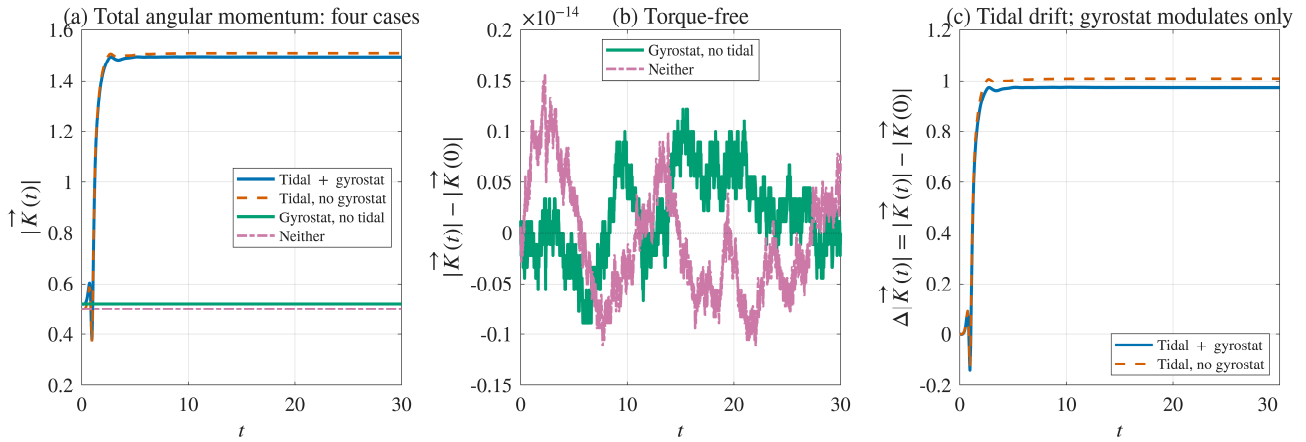


Fig. 8. Controlled four-case comparison isolating the source of the variation in $|K|$, obtained by switching the tidal torque and the gyrostatic bias independently on and off from a common Stage-1 terminal state.

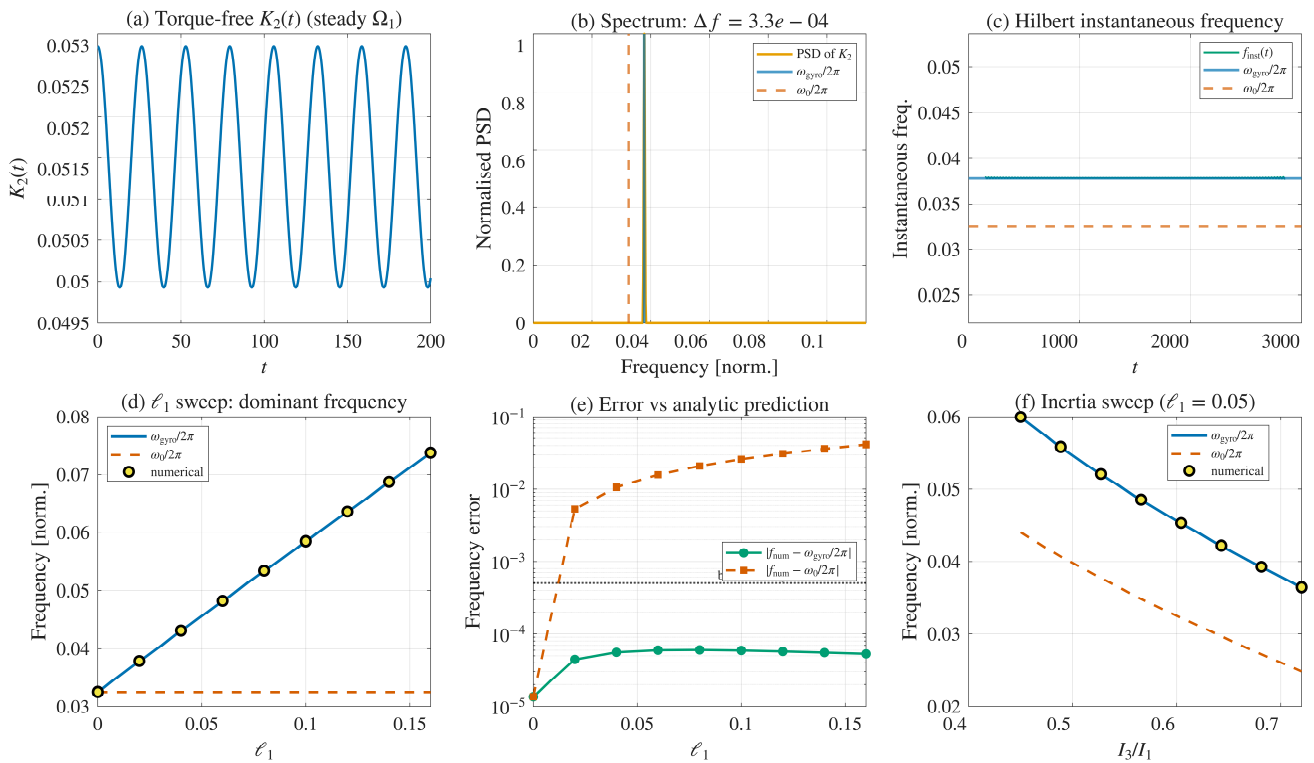


Fig. 9. Resolution-controlled validation of the gyrostatically shifted natural frequency (31) in its regime of derivation: the torque-free gyrostat spinning about the major axis, where Ω_1 is genuinely steady and the record can be made long enough for the spectral bin width to fall far below the shift.

Figure 10 reports the numerical diagnostics. Panel (a) confirms tolerance convergence of the Stage-2 solution; panel (b) shows that the Stage-1 residual in $|K|$ is integration error alone, falling with tightening tolerance; panel (c) verifies the angular-momentum theorem $|K|'(t) - |K|'(0) = \int_0^t K \cdot M dt'$ to a residual of order 10^{-12} ; panels (d) and (e) show that at the denominator zero-crossings the reduced estimate of Ω_3 diverges while the true Ω_3 remains bounded and the adaptive step size does not collapse; and panel (f) quantifies the phase-difference growth rate between the gyrostatic and classical solutions.

Finally, Fig. 11 re-integrates Stage 1 with unit-quaternion kinematics to confirm that the rapid excursions of ϕ and ψ at $\theta \rightarrow \pi$ are an artifact of the Euler chart. The quaternion and the body axes expressed in the inertial frame evolve smoothly through each passage, whereas the coordinate rate $|\dot{\phi}|$ spikes by several orders of magnitude relative to the bounded physical rate $|\dot{e}_1|$. This cross-check is continued over an interval longer than the nominal ejection phase so that several chart passages are captured in a single record; the mass-loss law (7) is extrapolated beyond its physical range in doing so, which is immaterial here because the conclusion concerns the coordinate chart alone and not the ejection model.

The energy budget requires separate comment since mechanical energy is conserved in neither stage: the moments of inertia decrease through mass ejection in Stage 1, and an external torque acts in Stage 2. The appropriate statements are therefore power balances rather than conservation laws, and both follow in closed form. For the flyby stage, with constant moments:

$$\frac{d}{dt} \left[\frac{1}{2} \underline{\Omega} \cdot I \underline{\Omega} \right] = \underline{\Omega} \times \underline{M}, \quad (37)$$

which is obtained by contracting Eq. (2) with $\underline{\Omega}$ and using the identities $\underline{\Omega} \cdot (\underline{\Omega} \times I \underline{\Omega}) = 0$ and $\underline{\Omega} \cdot (\underline{\Omega} \times \underline{\ell}) = 0$.



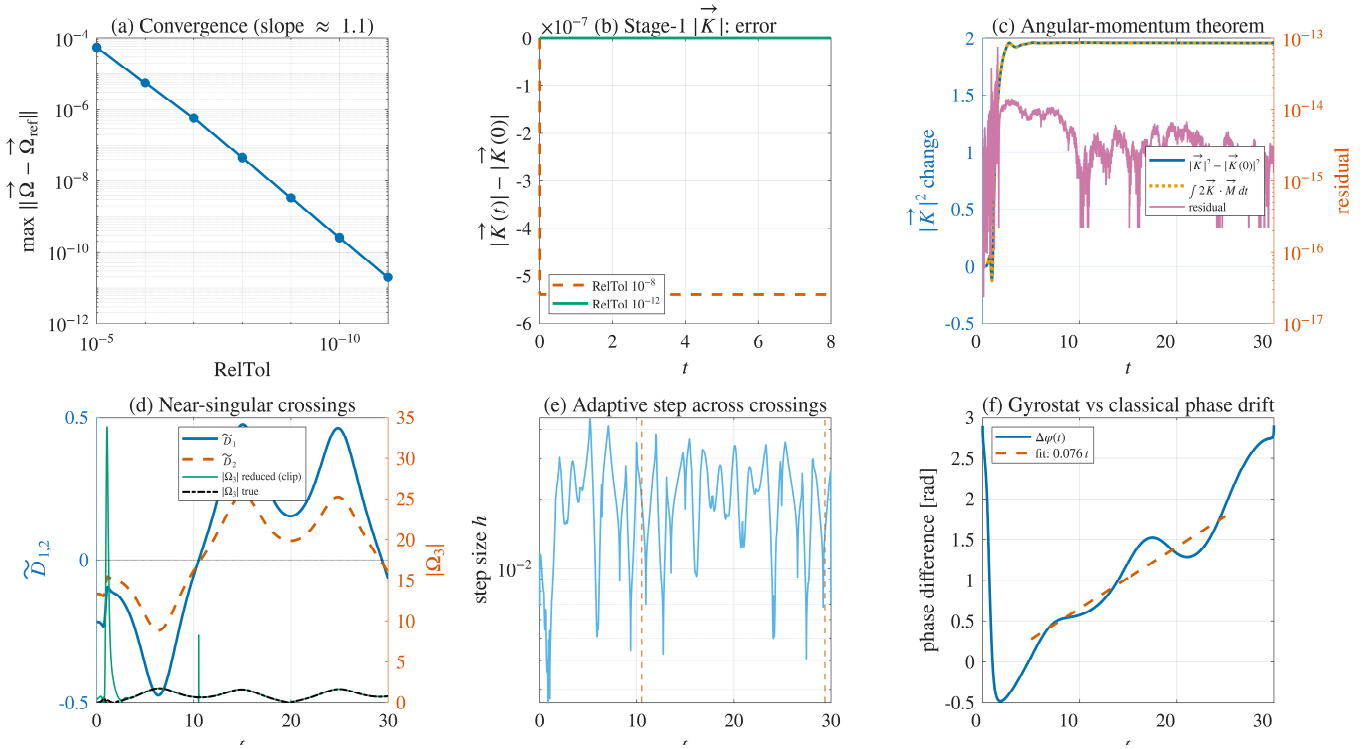


Fig. 10. Numerical diagnostics of the two-stage integration: convergence, conservation checks, angular momentum accounting, near-singular behavior, and phase-drift rate.

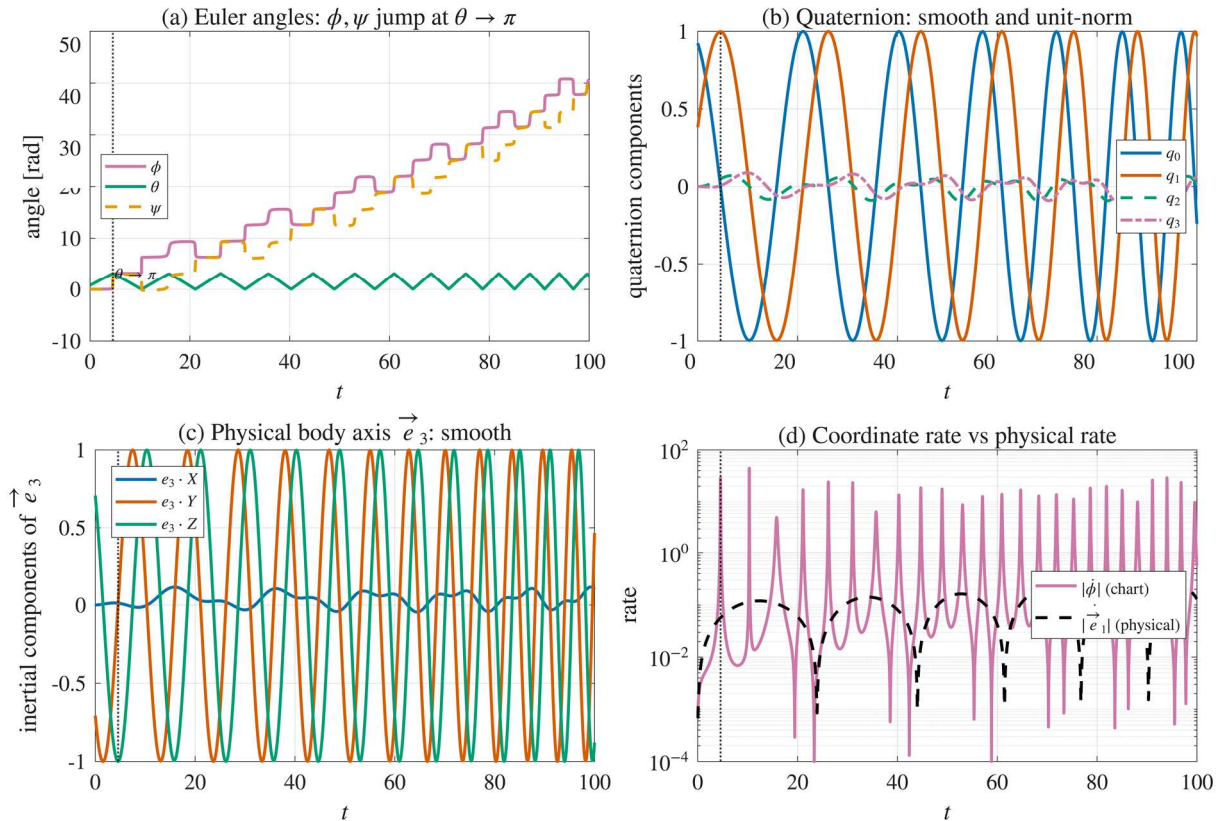


Fig. 11. Quaternion cross-check confirming that the rapid excursions of ϕ and ψ in Stage 1 are an artifact of the 3-1-3 Euler chart and not a feature of the physical motion. The torque-free Stage-1 dynamics is integrated with both orientation representations simultaneously.

The gyrostatic bias therefore drops out of the power balance exactly, in the same way and for the same reason that it drops out of $d|\vec{K}|^2/dt = 2\vec{K} \cdot \vec{M}$: the rotor redistributes energy and momentum among the components but supplies neither. For the ejection stage, with $\vec{M} = 0$ and $I_i = I_i(t)$, contraction of Eq. (2b) with $\vec{\Omega}$ gives instead:

$$\frac{d}{dt} \left[\frac{1}{2} \vec{\Omega} \cdot I(t) \vec{\Omega} \right] = -\frac{1}{2} \sum_{i=1}^3 \dot{I}_i \Omega_i^2 > 0, \quad (38)$$



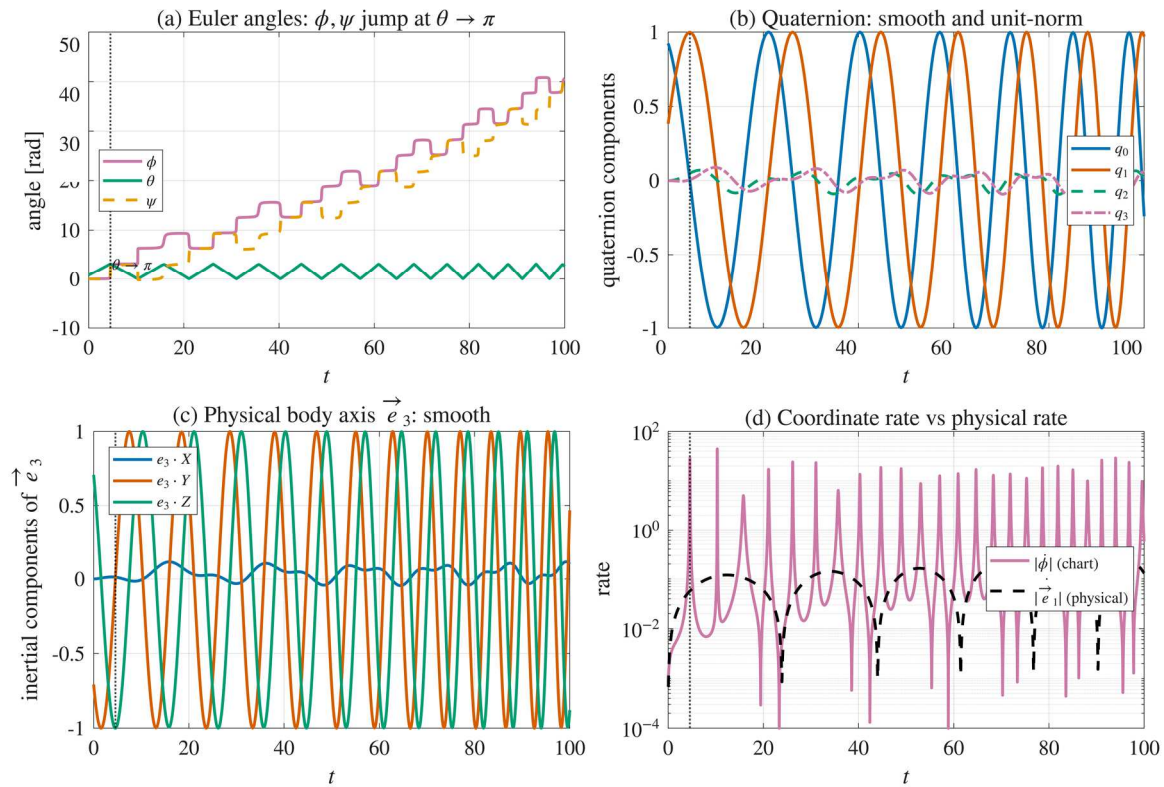


Fig. 12. Energy budget of the two-stage gyrostat. Mechanical energy is conserved in neither stage, so the appropriate statements are the power balances (37) and (38), each verified here with the power integral adjoined to the state vector so that both sides are produced by the same integrator.

the inequality following from $\dot{I}_i < 0$ under mass loss. The rotational energy of the retained body thus rises during ejection by precisely the amount demanded by the contraction of the inertia tensor at fixed $\mathbf{K}$, which is the energetic counterpart of the spin-up law (8). Figure 12 verifies both identities numerically. The residual of Eq. (37) falls to order 10^{-14} and that of Eq. (38) to order 10^{-12} , the slightly larger value in the ejection stage reflecting the integrable $t^{-1/4}$ cusp of $\dot{I}_i$ at the instant of impact; both fall further as the integration tolerance is tightened, confirming that the discrepancy is local truncation error and nothing else. Over the ejection phase the rotational energy of the retained body rises from 0.1250 to 0.1367, and over the flyby it rises from 0.1367 to 1.5329, the entire increment in the latter being delivered by the periapsis impulse. Panel (d) displays the two triple products $\underline{\Omega} \cdot (\underline{\Omega} \times \underline{\Omega})$ and $\underline{\Omega} \cdot (\underline{\Omega} \times \underline{\mathcal{L}})$ along the trajectory, each normalized by the magnitude of the terms whose cancellation produces it. Both remain below the machine epsilon $\varepsilon = 2.2 \times 10^{-16}$ throughout, attaining maxima of 7.8×10^{-17} and 1.3×10^{-16} , respectively, which confirms that the two identities used to derive Eq. (37) hold to rounding and that the rotor bias contributes no power.

6. Discussion

A semi-analytical investigation of the rotational dynamics of a gyrostatic rubble-pile asteroid during its close approach to a planet has been presented, extending the classical two-stage framework of [12] to account for the constant angular momentum contributions of internal rotors. The first stage, in which the asteroid is struck by a projectile far from the planet's Hill sphere [17], ejects rubble-pile material according to the cratering dynamics of [13], and the rate of rotation $\Omega_i(t)$ increases in accordance with Eq. (9). The gyrostatic terms introduce a first-order perturbation into the transverse angular velocity components that is absent from the classical treatment, but the spin-up law itself is shown to be independent of $\underline{\mathcal{L}}$ to leading order. The second stage is governed by the full gyrostat-extended Euler equations (2) with gravitational tidal torques. A key result is the derivation of Eq. (28). When Ω_1 is approximately constant, Eq. (28) reduces to the forced oscillator in Eq. (30) with the gyrostatically shifted natural frequency (31). As shown by Eq. (34), the gyrostatic correction to $\Delta(\omega^2)$ of Eq. (35) is positive throughout the spun-up phase and reverses sign only when the tidal forcing drives Ω_1 below the small negative threshold $-\ell_1 / (2I_1 - I_2 - I_3)$; the zero-crossings of the denominators $\tilde{\mathcal{D}}_{1,2}$ represent the most dynamically sensitive episodes of the rotational evolution.

Within the domain of validity set out above, the numerical results confirm the analytical predictions and additionally demonstrate: (i) a progressive phase drift between the gyrostatic and classical solutions, measured at 0.076 rad per unit time over $t \in [5, 25]$ in Fig. 10(f). This rate is not accounted for by the linear prediction $\Delta\omega$ of Eq. (36), which is itself of order 3.6×10^{-2} and changes sign within the same window; the discrepancy is expected, since Stage 2 lies well outside the small-oscillation regime in which Eq. (31) was derived, and the observed drift is a finite-amplitude effect for which the linearized frequency shift supplies only an order of magnitude; (ii) a measurable variation of the total angular momentum magnitude $|\mathbf{K}|$ produced by the external tidal torque and modulated by the gyrostatic coupling; (iii) the existence of near-singular zero-crossings of $\tilde{\mathcal{D}}_{1,2}$ at which the oscillatory character of the motion is temporarily disrupted; and (iv) a quantitative confirmation of ω_{gyro} in the regime in which it was derived, namely the torque-free major-axis spin of Figure 9, where the measured dominant frequency tracks $\omega_{\text{gyro}} / (2\pi)$ to within 6×10^{-5} across sweeps in ℓ_1 and in I_3 / I_1 , while departing from $\omega_0 / (2\pi)$ by as much as 4×10^{-2} . Among all admissible solutions, only those compatible with assumption (6) are physically meaningful within the present framework. The numerical results show that the ratio $|\Omega_{2,3} / \Omega_1|$, initialized at 10^{-3} , grows to $\sim 2 \times 10^{-1}$ by the end of Stage 1 and attains order unity within the first few time units of Stage 2. The semi-analytical formulae are therefore quantitative only over the earlier portion of each stage, and every result reported beyond that point rests on the fully nonlinear integration of Eqs. (2), (2a), (2b) and (4). This bounds the domain of the analytical description, but it does not compromise the frequency formula (31), which is validated independently in its own regime of derivation in Fig. 9.



7. Conclusion

The classical torque-free gyrostat and the tidally forced rigid-body asteroid model represent two well-studied limiting cases; the present work occupies and analyzes the intersection between them. By endowing the carrier body with constant internal rotor angular momentum and subjecting it simultaneously to rubble-pile mass ejection and gravitational tidal torques along a hyperbolic flyby trajectory, the analysis uncovers gyrostatic frequency shifting and near-singular spin transitions as the two primary dynamical consequences of the internal bias. The gyrostat-extended Euler equations (2) reduce to the classical formulation of [12] in the limit of a zero value and are therefore a proper generalization. The post-impact spin-up law (9) is independent of the gyrostatic bias ℓ_1 to leading order. The relation $\sin \psi \sin \theta = \text{const}$ is an exact first integral of the leading-order reduced Stage-1 kinematics obtained under assumption (6), rather than an invariant of the complete nonlinear system; within that reduction it is unaffected by the gyrostatic bias, being a consequence of the geometric relations (4) alone. The governing equation for $K_2(t)$ during Stage 2, derived as Eq. (28), is a second-order linear ODE with time-varying coefficients whose homogeneous part has a gyrostatically shifted natural frequency (31). The gyrostatic stiffening $\Delta(\omega^2) > 0$, with $\Delta(\omega^2)$ defined in Eq. (35), holds throughout the spun-up phase and reverses only when the tidal forcing drives Ω_1 below the small negative threshold $-\ell_1 / (2I_1 - I_2 - I_3)$. Numerical simulations confirm the analytical results within their stated domain of validity and additionally reveal a progressive gyrostatic phase drift; near-singular zero-crossings of the denominators $\mathcal{D}_{1,2}$ at which the reduced description fails while the full nonlinear system remains regular, and a measurable variation of $|\mathbf{k}|$ produced by the tidal torque and modulated by the gyrostatic bias. Physical disintegration of the asteroid is possible due to forced oscillations in the suggested scenario, requiring that internal stress states exceed material strength [8]. The gyrostatic modification of the oscillation frequency ω_{gyro} relative to ω_0 alters the resonance conditions under which such disintegration may occur and should therefore be accounted for in planetary-defense assessments.

Author Contributions

A.H. Elneklawy planned the scheme, developed the methodology and conceptualization, curated and validated the data, and contributed to visualization, original draft preparation, and editing; T.S. Amer supervised the work, conducted the investigation, contributed to methodology, data curation, and conceptualization, provided resources, and participated in validation and reviewing; I.M. Abady contributed to conceptualization, formal analysis, and validation. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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Conflict of Interest

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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