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Nonlinear Displacement Analysis of Functionally Graded Flexoelectric-Piezoelectric Nanobeams

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Abstract. The integration of flexoelectricity with piezoelectricity has recently attracted considerable interest due to its ability to enhance electromechanical coupling at the nanoscale. This paper presents a nonlinear finite element framework for the static analysis of functionally graded piezoelectric–flexoelectric nanobeams undergoing nonlinear displacements. The governing equations are derived using Hamilton's principle and incorporate nonlinear von Kármán strains, flexoelectric and piezoelectric effects, and power-law distributions of constituent materials along the nanobeam's longitudinal and transverse axes. The weak form is discretized using a mixed finite element formulation with the penalty method. Validation against homogeneous nanobeam cases demonstrates the accuracy and robustness of the proposed formulation. Comprehensive parametric studies are conducted to explore the influence of gradient indices, boundary conditions, and material composition on beam deflections. Results reveal that compliant polymer–ceramic combinations, such as PVDF–BaTiO₃, exhibit significantly larger deflections compared with purely ceramic configurations, highlighting the role of low-modulus constituents in enhancing flexibility. In contrast, stiffer ceramics such as SrTiO₃ and PbTiO₃ increase overall rigidity and reduce deflections. The developed framework provides new insights into the coupled electromechanical behavior of graded piezoelectric–flexoelectric nanobeams, offering valuable guidelines for the design of next-generation nanoelectromechanical systems. Applications include highly sensitive sensors, ultra-precise actuators, and nanoscale energy harvesters, where optimized material distributions can maximize electromechanical efficiency.

Keywords: Piezoelectricity; Flexoelectricity; Nanobeam; FGM; ANM; FEM.

1. Introduction

The rapid miniaturization of smart devices has driven a strong demand for advanced nanoelectromechanical systems (NEMS). In these microscopic structures, the coupling between mechanical strain gradients and electric polarization, known as the flexoelectric effect [1], has emerged as a crucial electromechanical phenomenon. Unlike piezoelectricity, which requires non-centrosymmetric crystal structures, flexoelectricity exists in all dielectric materials, making it highly relevant for the design of next-generation nanoelectromechanical systems [2–4]. When combined with piezoelectricity, flexoelectricity significantly enhances electromechanical responses, particularly in slender structures such as beams and plates [5, 6]. More specifically, nanobeams subjected to bending [7], buckling [8, 9], or vibrations [10, 11] can generate an electrical response through the combined effects of flexoelectricity and piezoelectricity. These coupled properties make them promising for applications such as sensors and actuators.

On the other hand, Functionally Graded Materials (FGMs) have been widely investigated for their ability to tailor material properties continuously through thickness and/or length using ceramics and polymers [12]. Such bidirectional functionally graded material is more complex than unidirectional ones. Recent advances in nanoscale fabrication, such as multi-material 3D nanoprinting, provides the technical foundation for producing such structures [13]. In the context of piezoelectric–flexoelectric nanobeams, FGMs allow optimization of stiffness, polarization, and induced electric potential. Previous studies have analyzed bending, vibration, and energy harvesting in graded nanobeams [14–17], demonstrating that material gradation strongly affects both static and dynamic performance. More recent investigations have shown that the position of the neutral axis [18], and the flexoelectric coefficient [19] play key roles in controlling size-dependent electromechanical coupling.

In addition, when the beam size decreases to the nanoscale, the combined effects of piezoelectricity and flexoelectricity become strongly size-dependent [5, 20]. Kvasov and Tagantsev [21] proposed a theoretical framework to estimate the flexoelectric coefficients using ab initio calculation, thereby making the simulation of the dynamic behavior of nanobeams more comprehensible. For instance, Deng and Shen [22] analyzed its role in energy harvesting, while Nguyen et al. [23] investigated its influence on natural frequencies. Yu et al. [15] examined the impact of gradient index parameters on bending, vibration, and polarization of functionally graded piezoelectric nanobeams. Similarly, Baroudi et al. [10] developed an analytical model that incorporates both bidirectional electric fields and strain gradients to predict the harvested voltage.

More recently, Liu et al. [16] studied the electromechanical coupling of FGM nanobeams under concentrated loads, assuming an exponential distribution of material properties along the thickness. Zheng et al. [17] investigated the mechanical behavior of FGM



flexoelectric nanobeams by combining flexoelectric theory with finite element modeling. Li et al. [19] showed that increasing the flexoelectric coefficient enhances the induced potential, with stronger effects as device size decreases. They also found that rotation gradients suppress beam deflection, with more pronounced influence for smaller aspect ratios.

Numerical methods have become the most effective tools to investigate flexoelectric solids having complex geometry, offering the flexibility to incorporate multiple-field couplings, size effects, and nonlinear behaviors. Abdollahi et al. [24] used a local maximum-entropy meshfree method to evaluate the flexoelectric effect in dielectric solids. On the other hand, mixed finite element formulations were used by Mao et al. [25] and implemented in Abaqus, enabling a consistent treatment of electromechanical fields while using compile problems such as plate with elliptical hole and cracked solids. Deng et al. [26] advanced this approach with a three-dimensional mixed finite element tailored to flexoelectricity, while Serrao and Kozinov [27] proposed a novel 3D mixed element for piezoelectric–flexoelectric materials, further improving numerical stability in capturing flexoelectric contribution for complex geometry solids. Ortigosa et al. [28] extended FEM into a micromorphic framework to perform topology optimization under finite strains, highlighting its potential for design applications. More application-oriented formulations have also emerged, such as the finite element approaches for flexoelectric nanobeam harvesters proposed by Sun et al. [29] and the study of functionally graded flexoelectric beams carried out by Zheng et al. [17]. Most recently, Abdelkhalek and Najjar [8] developed a systematic FEM formulation combined with the asymptotic numerical method, capable of tracing nonlinear responses and bifurcations in piezoelectric–flexoelectric nanobeams.

The nonlinear behavior of flexoelectric structures has attracted significant attention due to the combined influence of large displacements, geometric nonlinearities, and electromechanical coupling. In many cases researchers demonstrated that taking into account a moderate large displacement is enough to decently model micro and nanobeam structures. Ben Sassi and Najjar [30] modeled out-of-plane bending beams such as microcantilevers and microbridges to validate numerical models, demonstrating that geometric nonlinearities and mechanical coupling between axial and flexural behaviors are critical for accurately predicting the structure response. Further investigation into the dynamic response of such systems provided an experimental and analytical review of microbeams using von Kármán relations, establishing that moderately large initial deflections are safely approximated by these models [31, 32].

Early studies highlighted fundamental nonlinear effects in flexoelectric materials and structures [33], establishing the basis for theoretical models. More recently, nonlinear analyses of nanobeams have shown the importance of considering flexoelectricity in dynamic and static responses. For instance, Baroudi and Najjar [7] investigated nonlinear flexoelectric nanobeams under dynamic actuation, while Barati proposed models for nonlinear vibration and post-buckling responses of closed-circuit nanobeams, accounting for surface effects and imperfections [34, 35]. Additional research has emphasized nonlinearities in flexoelectric energy harvesters, where size dependency and force excitations significantly modify performance [36, 37]. Nonlinear flexoelectricity has also been examined in the context of energy harvesting and complex electromechanical responses [38, 39]. From a methodological standpoint, recent advances include systematic finite element approaches capable of capturing bifurcations and relatively large displacement regimes [8], and coupled nonlinear models based on modified continuum theories such as the Timoshenko beam or strain-gradient elasticity [40]. In parallel, investigations into the flexoelectric response of soft and biological systems [41] have expanded the understanding of geometric and material nonlinearities. Collectively, these contributions underline that nonlinear flexoelectric structures exhibit rich dynamics, where geometric effects and boundary conditions must be considered to accurately capture their behavior. Advanced continuation methods have been used to handle these nonlinear solutions. For example, Baroudi and Najjar [7] used a pseudo-arclength continuation method to extract the stable and unstable branches of a flexoelectric nanobeam. On the other hand, the Asymptotic Numerical Method (ANM) method has been widely applied in static and dynamic problems to efficiently detect bifurcations and instabilities, providing accurate results with fewer continuation steps [8, 42].

Despite these advances, most existing works have focused either on simplified analytical solutions or on homogeneous nanobeams. There remains a need for robust computational approaches capable of capturing nonlinear effects, nonlinear displacements, and coupled electromechanical fields in FGMs. This paper addresses this gap by developing a nonlinear finite element model for piezoelectric–flexoelectric nanobeams with arbitrary material gradation. The formulation accounts for von Kármán nonlinear strains, mixed boundary conditions, and spatially varying material distributions along both longitudinal and transverse directions. The main contribution of this work is to use a mixed finite element framework for graded piezoelectric–flexoelectric nanobeams [8] to investigate the influence of gradient indices, applied forces, and material combinations on static deflections.

Beyond the theoretical framework, bidirectional functionally graded nanobeams offer a practical solution to the efficiency loss of traditional piezoelectricity at the nanoscale by exploiting the dominance of gradient-dependent flexoelectricity. This framework enables the spatial optimization of electromechanical coupling in both the axial and transverse directions, allowing material properties to be matched to specific mechanical loads. Such design flexibility is particularly beneficial for energy harvesting, where maintaining high strain gradients over an optimized volume maximizes power density. These findings provide a blueprint for the strategic distribution of material phases in self-powered NEMS, offering essential guidelines for the development of ultra-precise medical sensors, high-stroke actuators, and robust micro-robotic components.

2. Modeling of the piezoelectric–flexoelectric nanobeam

The nanobeam considered in this study is illustrated in Figure 1. It is composed of a functionally graded material (FGM) distributed along the x and z axes. The FGM consists of two distinct constituents with piezoelectric–flexoelectric properties. The material distribution is assumed constant along the y direction. Two boundary condition configurations are examined: clamped–free (Figure 1b) and clamped–clamped (Figure 1c). The nanobeam is equipped with two thin compliant electrodes, one connected to ground potential and the other subjected to an external voltage V . The beam undergoes planar motion, experiencing bending and an induced axial deformation due to the boundary conditions. Its dimensions are defined by the length L , thickness h , and width b . The material properties are varied, and for a material property $\theta(x, z)$, the relationship between the values of the same property for the two constituents of the FGM, denoted by $\theta^{(1)}$ and $\theta^{(2)}$, is described as follows:

$$\theta(x, z) = \theta^{(1)} + (\theta^{(2)} - \theta^{(1)}) \left(\frac{x}{L} \right)^{n_x} \left(\frac{1}{2} + \frac{z}{h} \right)^{n_z} \quad (1)$$

where n_x and n_z are the power-law indices along the x and z directions, respectively, and $x \in [0, L]$ and $z \in [-\frac{h}{2}, \frac{h}{2}]$. The model assumes perfect bonding between materials [43].

2.1 Deformation and neutral axis

The nanobeam is modeled using the nonlinear Euler–Bernoulli beam theory. Denoting the transverse displacement by $w(x)$ and the axial displacement by $u(x)$, the only nonzero strain component is the axial strain ε . Including the nonlinear von Kármán strain,



it is expressed as [44]:

$$\varepsilon(x, z) = e - zw'' \quad (2)$$

where $e(x) = u' + \frac{1}{2}w'^2$ represents the midplane extension, and the prime denotes differentiation with respect to x . w'' is the second derivative of the transverse displacement and z is the material point position at the cross-section located at x . It is worth noting that the von Kármán strain-displacement relations used here are valid for moderately large deflections where rotations are small but non-negligible, allowing for the analysis of geometric nonlinearities within this range.

2.2 Enthalpy density

For relatively small electric fields, the materials can be assumed to exhibit linear electromechanical behavior. Under this assumption, the enthalpy density is expressed as follows [45]:

$$H = W^L(\mathbf{S}, \nabla \mathbf{S}, \mathbf{P}) - \frac{1}{2} \epsilon_0 \epsilon_r (\phi_{,x}^2 + \phi_{,z}^2) + P_x \phi_{,x} + P_z \phi_{,z} \quad (3)$$

Here, $\mathbf{S}$ denotes the Cauchy strain tensor, $\nabla \mathbf{S}$ the strain gradient tensor, $\mathbf{P} = (P_x(x, z), P_z(x, z))^T$ the polarization vector, and $\phi(x, z)$ the electric potential. The parameter ϵ corresponds to the electric permittivity.

Following material definitions commonly used in flexoelectric beam studies, the internal energy density W^L is expressed as [7, 46, 47]:

$$W^L = \frac{1}{2} a P_x^2 + \frac{1}{2} a P_z^2 + \frac{1}{2} c \varepsilon^2 + d \varepsilon P_z + f_1 \varepsilon_x P_x + f_3 \varepsilon_{,z} P_z + \frac{1}{2} g \varepsilon_{,x}^2 + \frac{1}{2} g \varepsilon_{,z}^2 \quad (4)$$

where $a(x, z) = \frac{1}{\epsilon_0(\epsilon_r(x, z) - 1)}$ is the dielectric susceptibility, $c(x, z)$ the longitudinal elastic modulus, $d(x, z)$ the piezoelectric constant, $f_1(x, z)$ and $f_3(x, z)$ the flexoelectric coefficients, and $g(x, z)$ the strain gradient coefficient given by $g = l_0^2 c$, with $l_0(x, z)$ denoting the material characteristic length scale. All these parameters follow the FGM variation described in Equation (1). Derivatives with respect to x and z are indicated by subscripts following the comma in their respective variables.

The proposed model assumes perfect bonding between the polymer and ceramic phases and uses an ideal, smooth mathematical curve for the material grading. In addition, the formulation allows for relatively large displacements and rotations by utilizing von Kármán nonlinear kinematics. The size-dependent behavior is captured using a strain gradient theory with two parameters along the x and z directions. We also assume linear constitutive equations, which means the results are primarily valid for moderate applied electric fields, and we restrict the electric polarization to the $x - z$ plane.

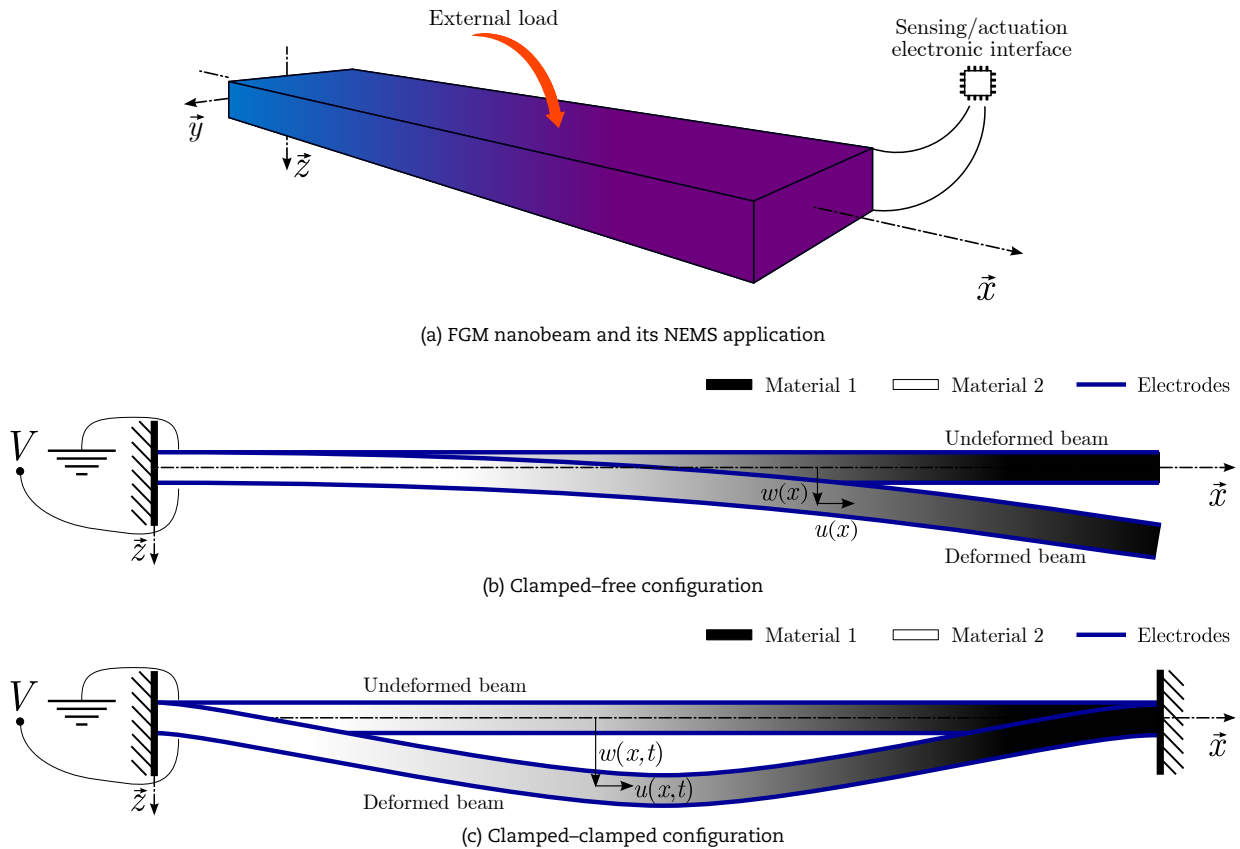


Fig. 1. FGM nanobeam composed of piezoelectric and flexoelectric materials.

2.3 Governing equations

The weak form of the problem is obtained from Hamilton's principle, which reads:

$$\int_{t_1}^{t_2} \left(\delta W_{nc} - \int_0^L \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta H \, dz \, dy \, dx \right) dt = 0 \quad (5)$$

where W_{nc} is the work of the non-conservative forces. h , b , and L are respectively the thickness, width and length of the nanobeam, as shown in Figures 1b and 1c. c is the functionally graded elastic modulus described by Equation (1).



The variation of the enthalpy density is decomposed into three parts:

$$\delta H = \delta H_1 + \delta H_2 + \delta H_3 \quad (6)$$

with

$$\begin{aligned} \delta H_1 &= T_{11} \delta \varepsilon + T_{111} \delta \varepsilon_{,x} + T_{113} \delta \varepsilon_{,z}, \\ \delta H_2 &= (P_x - \epsilon_0 \epsilon_r \phi_{,x}) \delta \phi_{,x} + (P_z - \epsilon_0 \epsilon_r \phi_{,z}) \delta \phi_{,z}, \\ \delta H_3 &= (\phi_{,x} + a P_x + f_1 \varepsilon_{,x}) \delta P_x + (\phi_{,z} + d \varepsilon + a P_z + f_3 \varepsilon_{,z}) \delta P_z, \end{aligned}$$

where T_{11} is the normal Cauchy stress, and T_{111} , T_{113} are higher-order stress components. They are expressed as:

$$T_{11}(x, z) = c \varepsilon + d P_z \quad (7)$$

$$T_{111}(x, z) = f_1 P_x + g \varepsilon_{,x} \quad (8)$$

$$T_{113}(x, z) = f_3 P_z + g \varepsilon_{,z} \quad (9)$$

The work done by an external transverse force F is given by:

$$\delta W_{nc} = b \int_0^L F \delta w \, dx \quad (10)$$

From Equations (5) and (6), the polarization components can be expressed in terms of the electric potential and displacement fields:

$$P_x(x, z) = -\frac{\phi_{,x}}{a} - \frac{f_1}{a} (e - z w'')' \quad (11)$$

$$P_z(x, z) = -\frac{\phi_{,z}}{a} - \frac{d}{a} (e - z w'') + \frac{f_3}{a} w'' \quad (12)$$

To determine the electric potential distribution ϕ , two assumptions are introduced: (i) due to symmetry about the y -axis, ϕ depends only on x and z ; (ii) its distribution across the thickness depends on both the applied voltage and the self-generated field from piezoelectric and flexoelectric effects [10, 48, 49]. Thus, ϕ is defined as:

$$\phi(x, z) = \cos\left(\frac{\pi}{h} z\right) \varphi(x) + \left(\frac{1}{2} + \frac{z}{h}\right) V \quad (13)$$

where V is the applied voltage and the lower electrode is grounded.

The assumed electric potential in Equation (13) is divided into two components. The term $\left(\frac{1}{2} + \frac{z}{h}\right) V$ represents the linear potential distribution imposed by the external voltage at the electrodes. The term $\cos\left(\frac{\pi}{h} z\right) \varphi(x)$ represents the internal potential induced by the piezoelectric and flexoelectric couplings. The cosine function is specifically chosen to satisfy the electrical boundary conditions, ensuring that the self-generated potential evaluates to zero at the highly conductive upper ($z = \frac{h}{2}$) and lower ($z = -\frac{h}{2}$) electrodes, while allowing the induced potential $\varphi(x)$ to vary along the longitudinal axis of the bending nanobeam.

Rewriting the normal Cauchy stress component T_{11} given by Equation (7), and using Equations (2), (12), and (13), we obtain:

$$T_{11} = c_1 e + \frac{d}{a} f_3 w'' - c_1 z w'' + c_6 \frac{\pi}{h} \sin\left(\frac{\pi}{h} z\right) \varphi - c_6 \frac{1}{h} V \quad (14)$$

Where the material related constants are replaced by the following quantities:

$$c_1 = c - \frac{d^2}{a}, \quad c_2 = \left(g - \frac{f_1^2}{a}\right), \quad c_3 = \left(g - \frac{f_3^2}{a}\right), \quad c_4 = \frac{f_3 d}{a}, \quad c_5 = \frac{f_1}{a}, \quad c_6 = \frac{d}{a}, \quad c_7 = \frac{f_3}{a}, \quad \kappa = \frac{1}{a} + \epsilon_0 \epsilon_r$$

Similarly, substituting into Equations (8) and (9), the higher-order stresses are expressed as:

$$T_{111} = c_2 e' - c_2 z w''' - c_5 \cos\left(\frac{\pi}{h} z\right) \varphi' \quad (15)$$

$$T_{113} = -c_4 e + c_4 z w'' - c_3 w'' - \frac{f_3}{a} \frac{1}{h} V + c_7 \frac{\pi}{h} \sin\left(\frac{\pi}{h} z\right) \varphi \quad (16)$$

Following the procedure proposed by Abdelkhalek and Najjar [8] to discretize the governing equation using a mixed finite element approach, and applying a Penalty method so that the classical Hermite element with a C^1 continuity can be suitable, we introduce the curvature χ defined as $\chi = w''$. As a result, the variation of the enthalpy density, defined in Equation (3), is given by



$$\begin{aligned}
& \int_0^L \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} [\delta H_1 + \delta H_2] dz dy dx \\
&= \int_0^L \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[T_{11} \delta \varepsilon + T_{111} \delta \varepsilon_{,x} + T_{113} \delta \varepsilon_{,z} + (P_x - \epsilon_0 \epsilon_r \phi_{,x}) \delta \phi_{,x} + (P_z - \epsilon_0 \epsilon_r \phi_{,z}) \delta \phi_{,z} \right] dz dy dx \\
&= b \int_0^L \int_{-\frac{b}{2}}^{\frac{b}{2}} \left[\left\{ c_1 e + (c_4 - c_1 z) w'' - \frac{1}{h} c_6 V + c_6 \frac{\pi}{h} \sin \left(\frac{\pi}{h} z \right) \varphi \right\} \delta e \right. \\
&\quad + \left\{ c_2 e' - z c_2 \chi' - c_5 \cos \left(\frac{\pi}{h} z \right) \varphi' \right\} \delta e' \\
&\quad + \left\{ (c_4 - z c_1) e + (z^2 c_1 + c_3 - 2 z c_4) w'' + \frac{1}{h} (c_7 + z c_6) V - (c_7 + z c_6) \frac{\pi}{h} \varphi \sin \left(\frac{\pi}{h} z \right) \right\} \delta w'' \\
&\quad + \left\{ -c_2 e' + z c_2 \chi' + c_5 \cos \left(\frac{\pi}{h} z \right) \varphi' \right\} z \delta \chi' \\
&\quad + \left\{ c_6 e - (c_7 + z c_6) w'' + \frac{1}{h} \kappa V - \kappa \frac{\pi}{h} \sin \left(\frac{\pi}{h} z \right) \varphi \right\} \frac{\pi}{h} \sin \left(\frac{\pi}{h} z \right) \delta \varphi \\
&\quad + \left\{ -c_5 e' + c_5 z \chi' - \kappa \cos \left(\frac{\pi}{h} z \right) \varphi' \right\} \cos \left(\frac{\pi}{h} z \right) \delta \varphi' \\
&\quad + \left\{ -c_6 e + (c_7 + z c_6) w'' - \frac{1}{h} \kappa V + \kappa \frac{\pi}{h} \sin \left(\frac{\pi}{h} z \right) \varphi \right\} \frac{1}{h} \delta V \\
&\quad \left. + \eta (\chi - w'') \delta (\chi - w'') \right] dz dx
\end{aligned} \tag{17}$$

In Equation (17), χ is the substitute variable for w'' , and η is the penalty parameter that enforces compatibility between them. For the boundary condition, we first introduce the following stress components defined by:

$$\begin{aligned}
M(x, t) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} (z T_1 + T_{13} - z T'_{11}) dz \\
\tilde{M}(x, t) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} z T_{11} dz \\
N(x, t) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} (T_1 - T'_{11}) dz \\
\tilde{N}(x, t) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} T_{11} dz
\end{aligned} \tag{18}$$

where $M(x, t)$ represent the bending moment, $N(x, t)$ the axial force and $\tilde{M}(x, t)$ and $\tilde{N}(x, t)$ are higher order moment and force components.

Therefore, Equation (17) can be rewritten as function of the stress components, then integrating by part and collecting the displacement and potential variables, we end up with:

$$\begin{aligned}
& \int_0^L \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} [\delta H_1 + \delta H_2] dz dy dx \\
&= \int_0^L \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[T_{11} \delta \varepsilon + T_{111} \delta \varepsilon_{,x} + T_{113} \delta \varepsilon_{,z} + (P_x - \epsilon_0 \epsilon_r \phi_{,x}) \delta \phi_{,x} + (P_z - \epsilon_0 \epsilon_r \phi_{,z}) \delta \phi_{,z} \right] dz dy dx \\
&= b \int_0^L \left[-N' \delta u - (M'' + N' w' + N w'') \delta w + \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\left(-c_5 e' + z c_5 w''' - \kappa \cos \left(\frac{\pi}{h} z \right) \varphi' \right)' \cos \left(\frac{\pi}{h} z \right) + (c_6 e - (c_7 + z c_6) w'' \right. \right. \right. \\
&\quad + \left. \left. \frac{1}{h} \kappa V - \kappa \frac{\pi}{h} \sin \left(\frac{\pi}{h} z \right) \varphi \right) \frac{\pi}{h} \sin \left(\frac{\pi}{h} z \right) \right] \delta \varphi + \left(-c_6 e + (c_7 + z c_6) w'' - \frac{1}{h} \kappa V + \kappa \frac{\pi}{h} \sin \left(\frac{\pi}{h} z \right) \varphi \right) \frac{1}{h} \delta V \right] dz \Big] dx \\
&\quad + [N \delta u]_0^L + [\tilde{N} \delta u']_0^L + [(N w' - M') \delta w]_0^L + [M \delta w']_0^L - [\tilde{M} \delta w'']_0^L \\
&\quad + \left[b \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(-c_5 e' + z c_5 w''' - \kappa \cos \left(\frac{\pi}{h} z \right) \varphi' \right) \cos \left(\frac{\pi}{h} z \right) dz \delta \varphi \right]_0^L
\end{aligned} \tag{19}$$

$$\begin{aligned}
M(x, t) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} [(z c_1 - c_4) \varepsilon + (z c_4 - c_3) w'' - z (c_2 \varepsilon_{,x})' + z (c_5 \phi_{,x})' - (z c_6 + c_7) \phi_{,z}] dz \\
\tilde{M}(x, t) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} z [-c_5 \phi_{,x} + c_2 \varepsilon_{,x}] dz \\
N(x, t) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} [c_1 \varepsilon + c_4 w'' - (c_2 \varepsilon_{,x})' + (c_5 \phi_{,x})' - c_6 \phi_{,z}] dz \\
\tilde{N}(x, t) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} [-c_5 \phi_{,x} + c_2 \varepsilon_{,x}] dz
\end{aligned} \tag{20}$$



The associated boundary conditions for the two configurations are given as follows:

Clamped-free:

$$\begin{aligned} \text{At } x = 0 : \quad & u = 0, u' = 0, w = 0, w' = 0, w'' = 0, \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(-c_5 e' + z c_5 w''' - \kappa \cos\left(\frac{\pi}{h} z\right) \varphi' \right) \cos\left(\frac{\pi}{h} z\right) dz = 0 \\ \text{At } x = L : \quad & N = 0, \tilde{N} = 0, M = 0, M' = 0, \tilde{M} = 0, \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(-c_5 e' + z c_5 w''' - \kappa \cos\left(\frac{\pi}{h} z\right) \varphi' \right) \cos\left(\frac{\pi}{h} z\right) dz = 0 \end{aligned} \quad (21)$$

Clamped-clamped:

$$\text{At } x = 0, L : \quad u = 0, u' = 0, w = 0, w' = 0, w'' = 0, \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(-c_5 e' + z c_5 w''' - \kappa \cos\left(\frac{\pi}{h} z\right) \varphi' \right) \cos\left(\frac{\pi}{h} z\right) dz = 0$$

These conditions ensure the proper mechanical constraints and electrical boundary requirements at the electrodes for each beam configuration.

2.4 Discretization technique and solution algorithm

Merging Equations (10) and (17) into the global energy functional defined by Hamilton's principle in Equation (5) establishes the coupled electromechanical equilibrium. By substituting the material parameters with their respective power-law distributions defined in Equation (1), the local constitutive relations are explicitly linked to the spatial coordinates.

The resulting weak form is discretized over the longitudinal x -domain using a high-order mixed finite element scheme. To ensure C^1 continuity for the displacement fields, as requested by the presence of strain gradient and flexoelectric terms, Hermite cubic interpolation functions are employed. Due to the inherent geometric nonlinearity introduced by the von Kármán strains, the solution of the discretized system requires a robust iterative strategy. In this work, the systematic Asymptotic Numerical Method (sANM) is applied to trace the nonlinear equilibrium path [8]. Unlike traditional Newton-Raphson techniques, the sANM transforms the nonlinear problem into a series of linear sub-problems using a Taylor series expansion of the variables. This approach offers superior stability in capturing the response of the nanobeam without the convergence issues typically associated with high-gradient flexoelectric coupling.

The discretization technique and the general solution algorithm based on the sANM employed in this study follow the methodology detailed in our previous work [8].

Fundamentally, the ANM seeks the solution branch (U, λ) as a power series in the step parameter a (arc length), truncated at order n :

$$U = U_0 + \sum_{p=1}^n a^p U_p, \quad \lambda = \lambda_0 + \sum_{p=1}^n a^p \lambda_p, \quad (22)$$

where λ is the load parameter. To systematically process the equations, every term of the weak form is recast into a generic bilinear-quadratic form:

$$\int_L (\alpha + \beta \gamma) \delta(X + YZ) dx, \quad (23)$$

where $\alpha, \beta, \gamma, X, Y, Z$ are unknowns of the problem (or linear combinations thereof). The variation expands as

$$\delta(X + YZ) = \delta X + Z \delta Y + Y \delta Z, \quad (24)$$

which shows that, within the ANM, the variable X does not require a series expansion. Likewise, every other scalar variable of the generic form is expanded into a power series:

$$\alpha = \alpha_0 + \sum_{p=1}^n a^p \alpha_p, \quad \beta = \beta_0 + \sum_{p=1}^n a^p \beta_p, \quad \gamma = \gamma_0 + \sum_{p=1}^n a^p \gamma_p, \quad Y = Y_0 + \sum_{p=1}^n a^p Y_p, \quad Z = Z_0 + \sum_{p=1}^n a^p Z_p. \quad (25)$$

For the spatial discretization, on an element of length l_e with local coordinate $r \in [-1, 1]$, the fields are interpolated using the vector of Hermite functions $\mathbf{H}$:

$$\mathbf{H} = \begin{cases} H_1 = \frac{1}{4}(r-1)^2(r+2) \\ H_2 = \frac{l_e}{8}(r-1)^2(r+1) \\ H_3 = -\frac{1}{4}(r+1)^2(r-2) \\ H_4 = \frac{l_e}{8}(r+1)^2(r-1) \end{cases} \quad (26)$$

Consequently, any quantity q_p appearing at order p of the series is discretized by a vector of Hermite-function derivatives ψ_q and the corresponding elementary nodal vector:

$$\begin{aligned} \alpha_p &= \psi_\alpha^T \alpha_p^e, & \beta_p &= \psi_\beta^T \beta_p^e, & \gamma_p &= \psi_\gamma^T \gamma_p^e, \\ X_p &= \psi_X^T X_p^e, & Y_p &= \psi_Y^T Y_p^e, & Z_p &= \psi_Z^T Z_p^e. \end{aligned} \quad (27)$$

Finally, substituting Eq. 25 into Eq. 23 and collecting the powers of a yields a cascade of linear problems, one per order p . The overall solution algorithm and branch following procedure are summarized in Figure 2.

3. Response of the nanobeam for different FGM parameters

In this section, the static response of the nanobeam is investigated for different combinations of FGMs. The effect of material distribution parameters, loading, and boundary conditions is examined through numerical simulations. Material and geometrical properties are summarized in Tables 1 and 2.

Table 2. Geometrical properties of the FGM nanobeam.

Length L (nm)	Thickness h (nm)	Width b (nm)
400	20	20



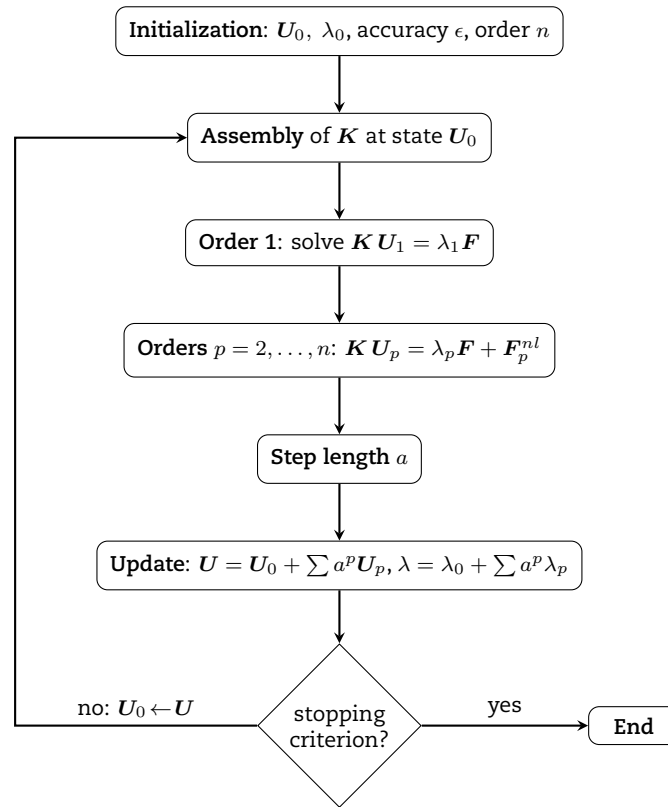


Fig. 2. ANM continuation loop.

Table 1. The material properties used for the FGM nanobeam.

	BaTiO ₃ [6]	SrTiO ₃ [50, 51]	PbTiO ₃ [52–55]	PVDF [10, 56, 57]
c (GPa)	131	348	237	3.7
d (V/m)	1.87×10^8	-0.755×10^8	3.1×10^8	-9.29×10^8
f_1 (V)	0	0	0	–200
f_3 (V)	5	–3	5	–10
l_0 (nm)	3	5	3	12
ϵ_r	1430	300	340	10
ϵ_0 (C ² /m ² /N)	8.85×10^{-12}			

3.1 Comparison with other methods

To validate the proposed formulation, the homogeneous case is first considered for BaTiO₃, with results compared against existing studies in the literature. Both clamped–free and clamped–clamped boundary conditions are tested using material properties in Table 1, and the beam dimensions of Table 2. The nondimensional deflections obtained from the present model are in good agreement with previously published data [6, 7, 49], as shown in Figure 3 for two different boundary condition configurations. This confirms that the formulation correctly captures the behavior of homogeneous nanobeams, with or without the strain gradient effect ($l_0 = 0$ and $l_0 \neq 0$) for both clamped–free (Figure 3a) and clamped–clamped (Figure 3b) configurations. The deviation between the models highlights the physical impact of the size-dependent stiffening effect. Disregarding l_0 reduces the formulation to classical continuum mechanics, which neglects microstructural interactions and consequently overestimates the nanobeam’s flexibility.

3.2 Analysis of the effect of FGM composition

Before analyzing FGM cases, the behavior of each homogeneous constituent material is first examined. Figure 4 shows the static deflections of nanobeams made from BaTiO₃, SrTiO₃, PbTiO₃, and PVDF under an applied external force $F = -1$ nN and a voltage $V = -0.1$ V, and material properties given by Table 1. As expected, for clamped–free (Figure 4a) and clamped–clamped (Figure 4b) configurations, PVDF exhibits the largest deflection due to its low stiffness, while the stiffer ceramics produce smaller deflections. This comparison highlights the dominant role of elastic modulus in governing the transverse response.

Next, the role of the power-law exponents n_x and n_z governing the FGM distribution is analyzed. Their variation, between 0 and 5, determines the spatial distribution of constituent materials along the longitudinal and transverse directions, as illustrated in Figure 5. Darker colors represent higher proportions of material 1, while lighter colors correspond to the other constituent.

3.2.1 Validity Range of the von Kármán Model

To ensure that the numerical results presented in this study remain consistent with the assumptions of the von Kármán nonlinear beam theory. While this theory accounts for moderate rotations, it assumes that the strains remain small [58]. In Figure 6 we monitor the maximum axial strain developing in the nanobeam during the loading process. It is calculated at each equilibrium



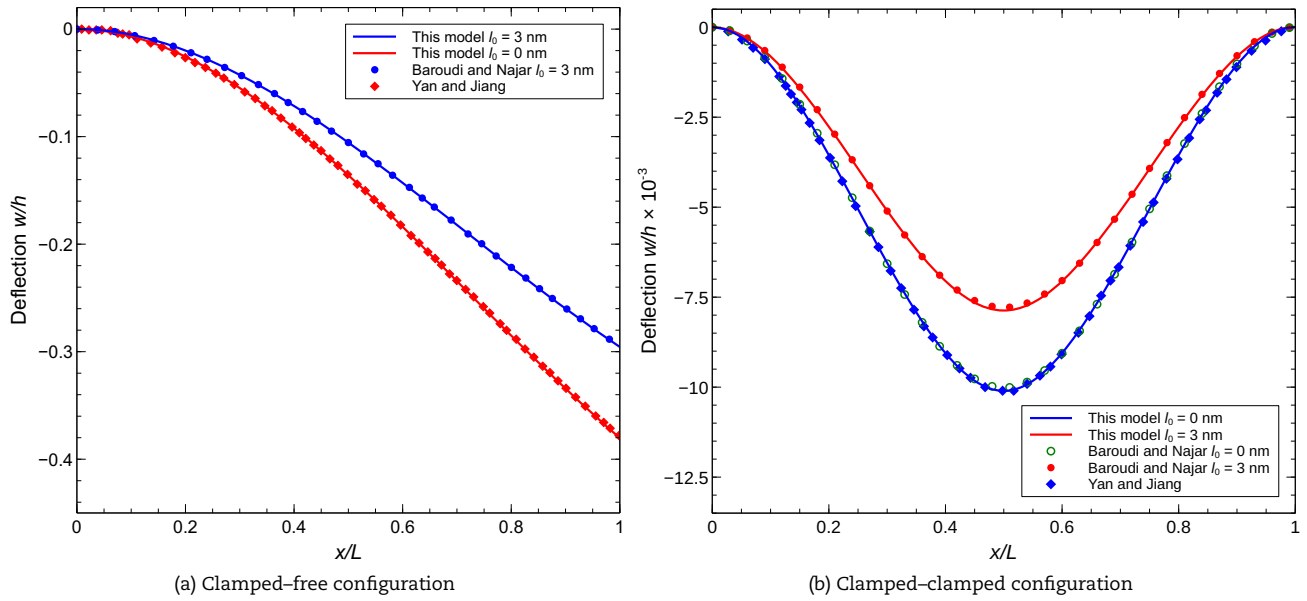


Fig. 3. Nondimensional transverse deflection of a BaTiO₃ nanobeam, compared with published results. $F = -1$ nN and $V = -0.1$ V.

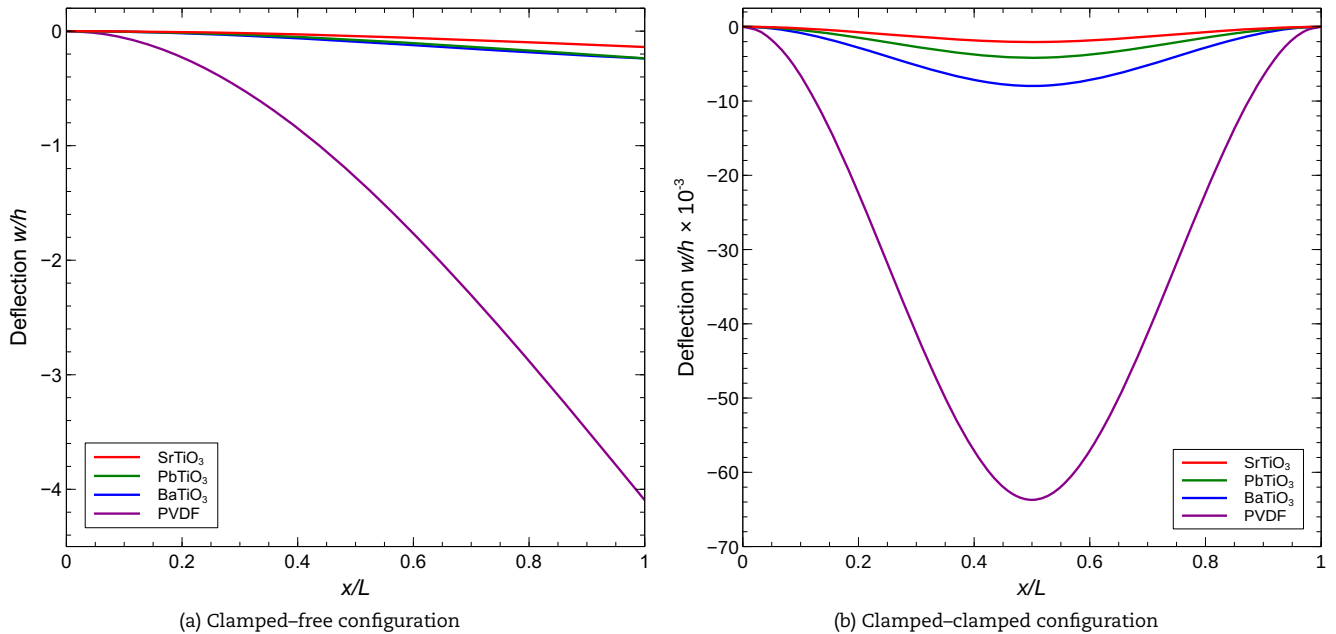


Fig. 4. Nondimensional transverse deflection of nanobeams with different homogeneous materials. $F = -1$ nN and $V = -0.1$ V.

point as

$$\varepsilon_{\max} = \max_{\substack{0 \leq x \leq L \\ -\frac{h}{2} \leq z \leq \frac{h}{2}}} \left| u' + \frac{1}{2}(w')^2 - zw'' \right|$$

Figure 6 illustrates the evolution of $\varepsilon_{\max}$ as a function of the power-law exponents n_x and n_z for both clamped-clamped and clamped-free boundary conditions. The analysis covers a wide range of PVDF–BaTiO₃ material distributions, with the transverse power-law exponent n_z varying from 0 to 5 and the longitudinal exponent n_x taking values of 0, 2, and 5. Due to the high flexibility of the polymer phase, configurations rich in PVDF naturally generate the largest strains between the treated cases in our analysis. Nevertheless, the calculated strain is expected to remain below 5%. This threshold is supported by Neukirch et al. [59], who established that using the von Kármán strain model for a beam with a slenderness ratio of 20, the maximum displacement must not exceed four times the beam thickness. As observed in Figure 6a, the maximum strain values for the clamped-free configuration stay below 0.05 (5%) even in the worst-case scenario ($n_x = 5$ and $n_z = 5$), which represents a nanobeam composed almost entirely of PVDF. For the clamped-clamped configuration shown in Figure 6b, the strains are considerably lower due to the added boundary constraints that minimize deflection. Ultimately, while adjusting the power-law exponents directly influences the structural stiffness, these results confirm that the small strain assumption ($\varepsilon_{\max} \ll 1$) holds true across the entire design space, validating the use of the von Kármán model.



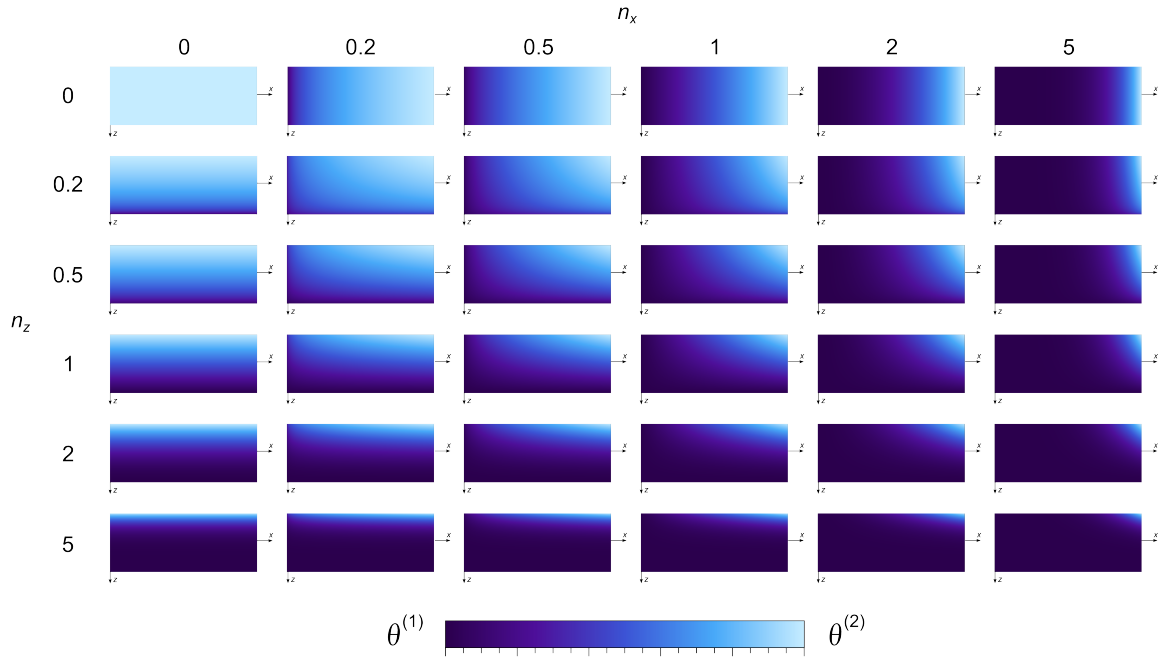


Fig. 5. Material property distribution over the nanobeam's xz -axis, for different power-law exponents n_x and n_z .

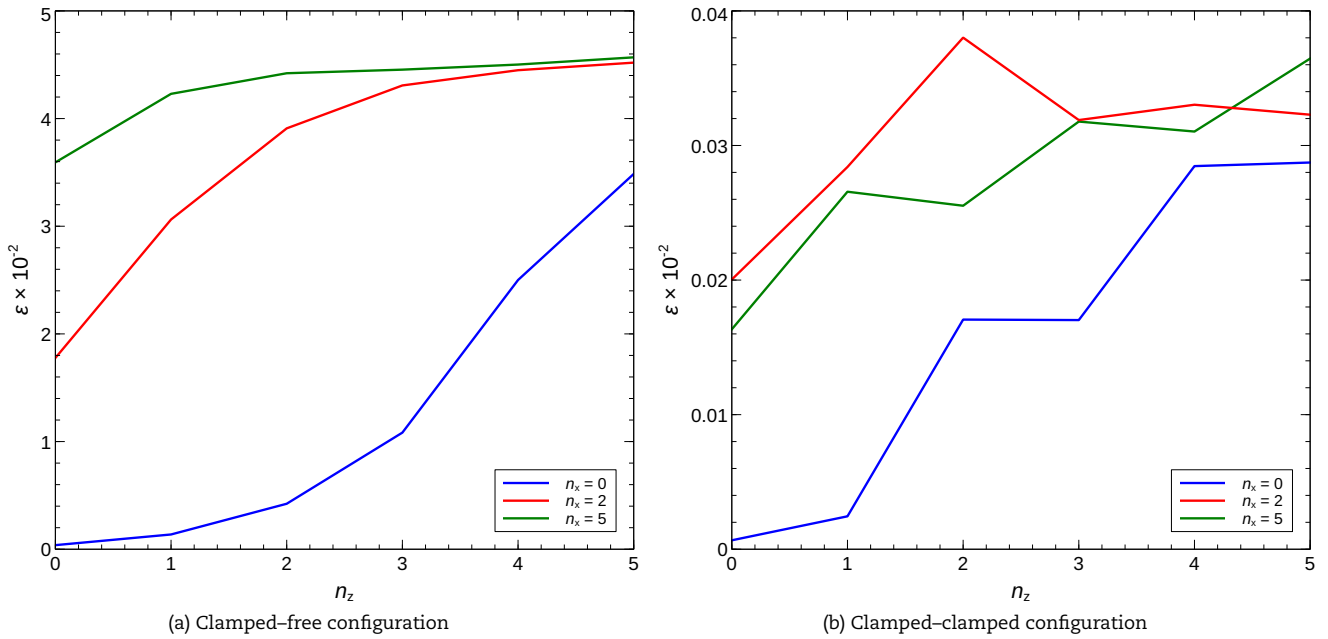


Fig. 6. Validation of the small strain assumption: Variation of the maximum axial strain versus different power-law exponents n_x and n_z .

3.2.2 Variation in the transverse direction using n_z

We first investigate the influence of the transverse power-law exponent n_z , which governs material gradation across the beam thickness. In Figure 7 an FGM composed by a combination of SrTiO_3 – BaTiO_3 materials with different values of the power-law exponent n_z varied between 0 and 5. The different cases treated here are those corresponding to the first column of material distribution shown in Figure 5. For the clamped-free configuration the response is given by Figure 7a where it is clear that the beam deflection is larger when the material is composed only by BaTiO_3 ($n_x = 0$ and $n_z = 0$) and it is less important when the major part of the beam is composed of SrTiO_3 ($n_x = 0$ and $n_z = 5$). This is expected since for this case the elastic modulus of material 2 (BaTiO_3) is higher than material 1 (SrTiO_3), therefore a larger deflection is obtained. As a result, for the set of external force and voltage applied to the nanobeam, the deflection is dominated only by the elasticity of the materials. The same comments can be done for the clamped-clamped nanobeam deflection presented in Figure 7b, since the same material is used.

For PbTiO_3 – BaTiO_3 combinations, the deflections remain close for all values of n_z , reflecting the similarity of their stiffness. The clamped-free case, shown in Figure 8a, illustrate that the response is only slightly changed even for large values of n_z . It is also noted that for $n_z = 0$ the curvature near the nanobeam tip is higher than the other cases. In fact, the flexoelectric effect with a negative voltage V can be assumed to be equivalent to a positive bending moment applied at the nanobeam's tip for the clamped-free case [6]. Therefore, it is concluded here that the introduction of material 1 reduces the effect of the flexoelectric moment so that the tip displacement due to the applied force and the indirect piezoelectric effect is no longer mitigated by the tip moment even though the effective stiffness of the nanobeam is increased due to material 1. For the clamped-clamped case, shown in Figure 8b,



the increase of the value of n_z from 0 to 5 reduces the deflection of the nanobeam indicating that its effective stiffness is increased by the introduction of material 1 (PbTiO₃).

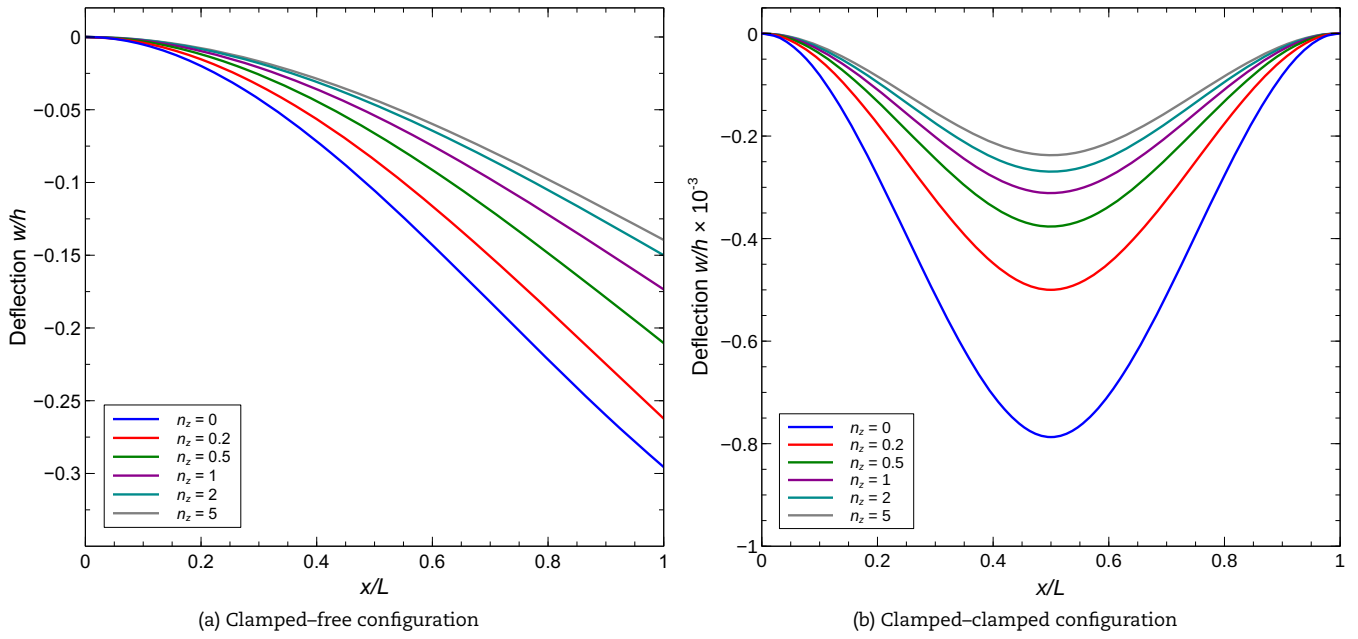


Fig. 7. Deflection of a SrTiO₃-BaTiO₃ FGM nanobeam for different n_z . $n_x = 0$, $F = -1$ nN, $V = -0.1$ V.

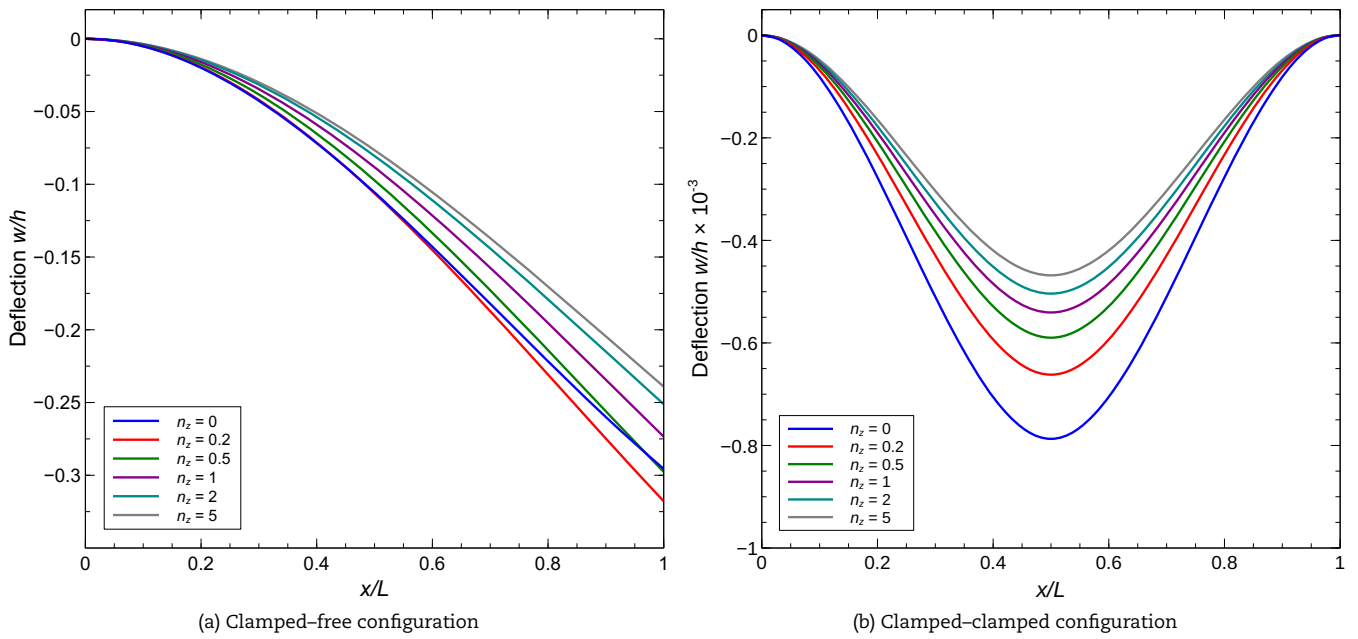


Fig. 8. Deflection of a PbTiO₃-BaTiO₃ FGM nanobeam for different n_z . $n_x = 0$, $F = -1$ nN, $V = -0.1$ V.

In Figure 9, PVDF is used as material 1 in the FGM. Since the elastic modulus of PVDF is much lower than BaTiO₃, it is expected that the deflection of the nanobeam substantially increase in both clamped-free and clamped-clamped cases. Figures 9a and 9b confirm these expectations and show that the deflection is even more noticeable as the value of n_z is higher, demonstrating how PVDF drastically enhance flexibility. For the clamped-clamped case, it is remarkable that the symmetry of the solution is maintained as long as the value of n_x is zero.

3.2.3 Variation in the longitudinal direction using n_x

In this section the value of power-law exponent n_x is varied while maintaining $n_z = 0$. This variation corresponds to the first row of the material distribution over the nanobeam illustrated by Figure 5 where it is shown that, if $n_x \neq 0$, the FGM removes the symmetry of the nanobeam's material in the longitudinal direction, especially for the clamped-clamped case. For the FGM whose constituent are respectively SrTiO₃ and BaTiO₃, the deflections of the clamped-free nanobeam are depicted by Figure 10a. As expected the tip displacement is reduced as n_x is increased. The same behavior is shown in Figure 10b for the clamped-clamped case. In addition, an asymmetric solution is clearly obtained when $n_x \geq 2$ denoting that the effect of adding SrTiO₃ as material 1, not only increases the effective stiffness of the nanobeam but also change the direction of the transverse deflection, in specific locations



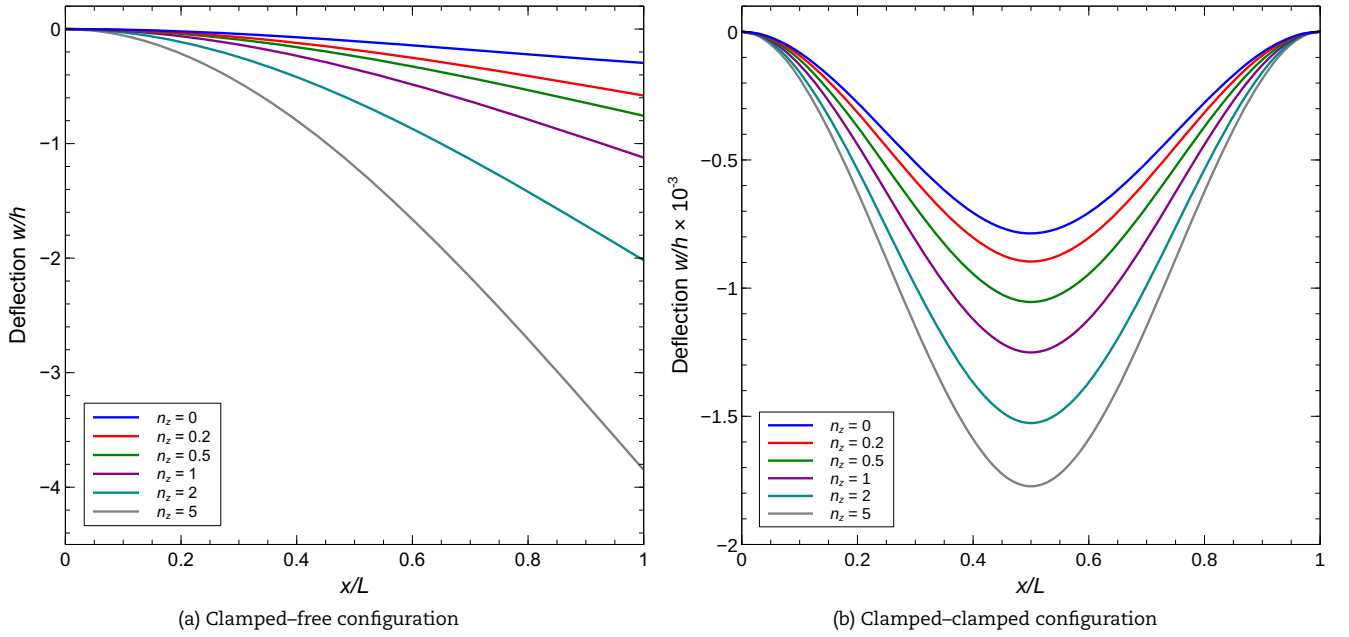


Fig. 9. Deflection of a PVDF-BaTiO₃ FGM nanobeam for different n_z . $n_x = 0$, $F = -1$ nN, $V = -0.1$ V.

of the nanobeam, from negative to positive. The former effect can be related to the inverse sign of the flexoelectric coefficient between the two constituents of the FGM.

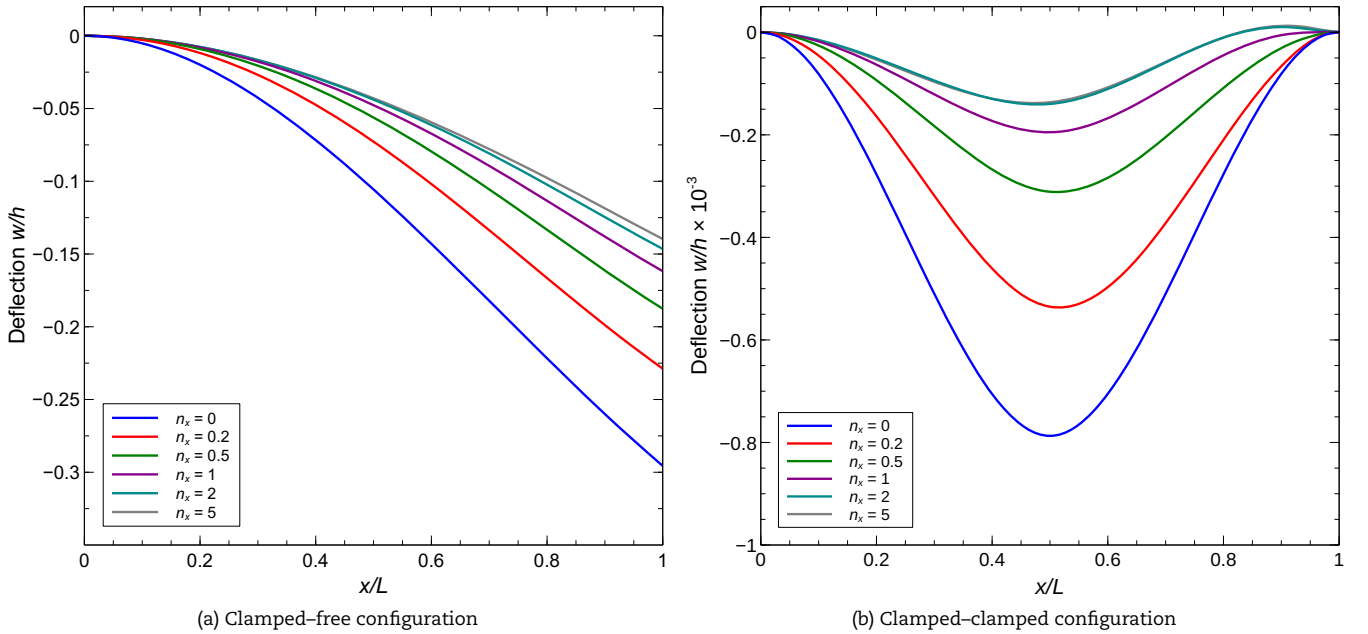


Fig. 10. Deflection of a SrTiO₃-BaTiO₃ FGM nanobeam for different n_x . $n_z = 0$, $F = -1$ nN, $V = -0.1$ V.

The same set of power-law exponent variation is applied to the FGM formed by PbTiO₃-BaTiO₃. The results are shown in Figure 11 for two different boundary conditions illustrate the response of the corresponding nanobeam. The observed variations of the nanobeam's deflections are qualitatively similar to the previous case (Figure 10) with smaller amplitude changes. For the clamped-free case depicted by Figure 11a, the simulated deflections are very close to each other due to the fact that both material have relatively close physical properties compared to the other cases. The clamped-clamped case shown by Figure 11b also depict relatively smaller sensitivity of the FGM to variations of the power-law exponent n_x , compared to the previous case. One can conclude here that the clamped-clamped boundary condition case is more sensitive to the clamped-free case when n_x is varied.

For the FGM PVDF-BaTiO₃, the corresponding simulation results are illustrated by Figure 12. For the clamped-free boundary conditions in Figure 12a, the bending amplitude is drastically increase as the power-law exponent n_x is increased. This is due to very low elastic modulus of the PVDF introduced near the clamped side of the nanobeam and progressively moving to the free side as n_x is increased (Figure 5), which increases further the bending response. For the clamped-clamped boundary condition illustrated by Figure 12b, the situation is different since a change of the power-law exponent n_x from 0 to 0.2 reduces the bending amplitude of the nanobeam especially around its right side. However, a further increase of n_x largely increases the deflection. It is concluded here that two competing effects are taking place: one is related to the effective stiffness of the FGM due to the elastic modulus difference, and the second is related to the piezoelectric-flexoelectric effect that has an opposite effect because of the sign change of the piezoelectric and flexoelectric coefficients of material 1 and 2.



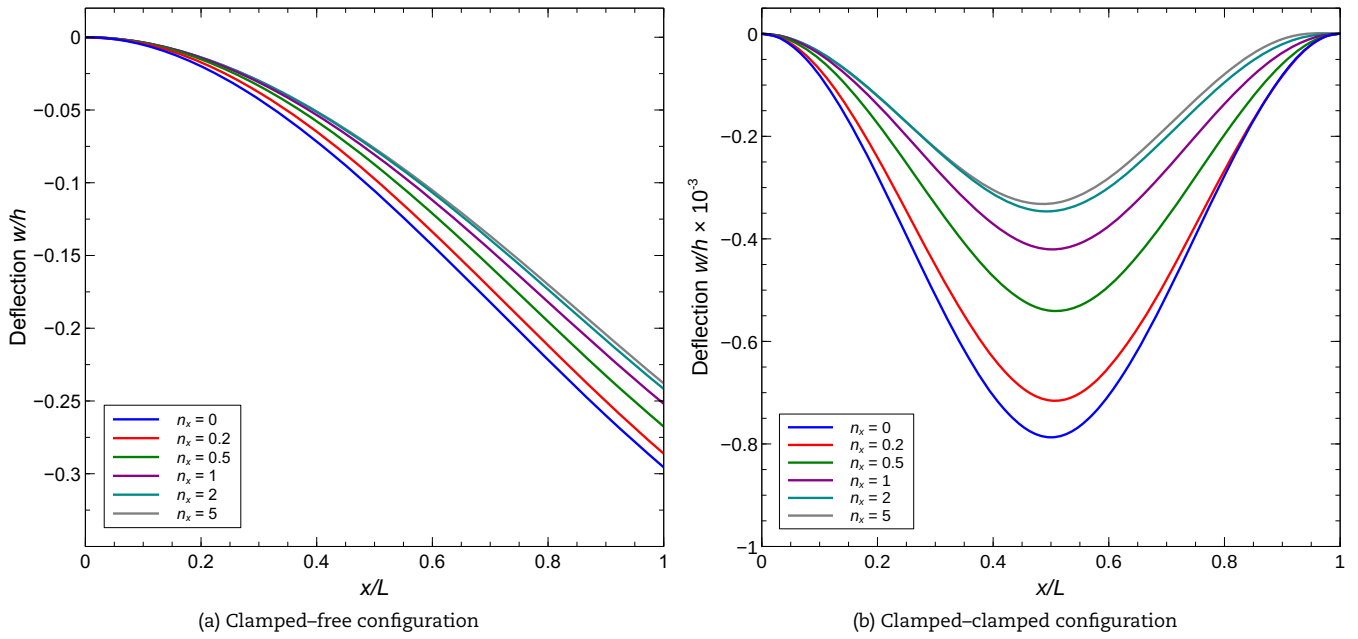


Fig. 11. Deflection of a PbTiO₃-BaTiO₃ FGM nanobeam for different n_x . $n_z = 0$, $F = -1$ nN, $V = -0.1$ V.

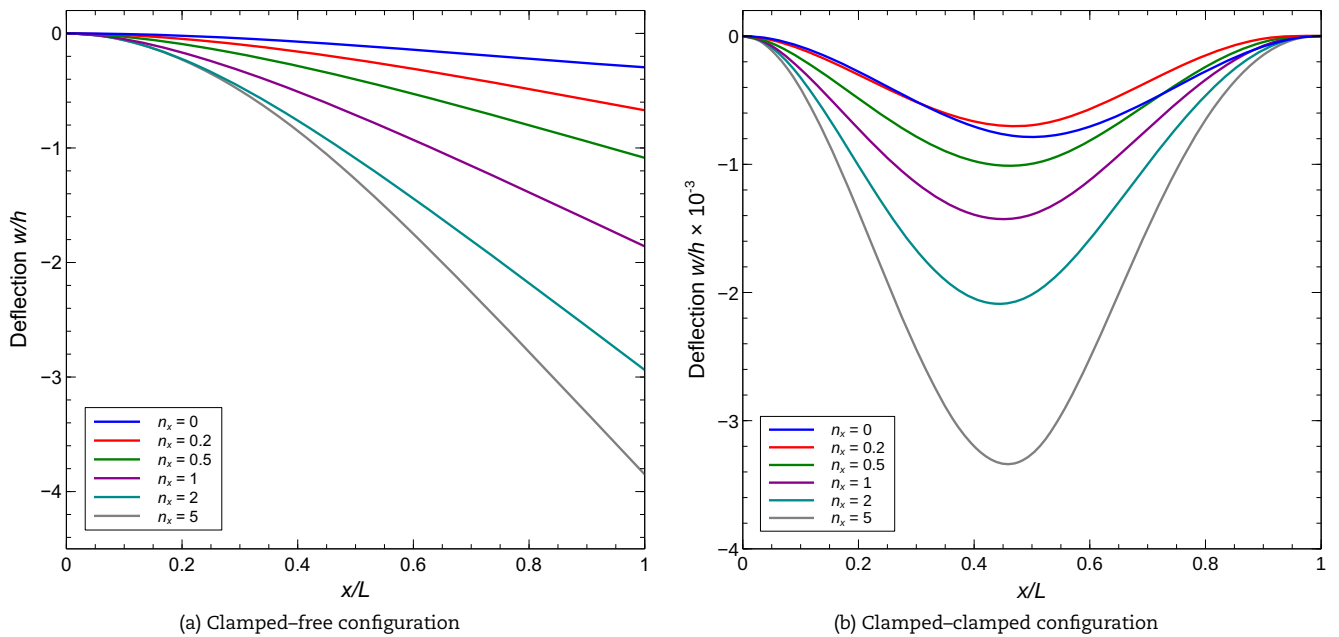


Fig. 12. Deflection of a PVDF-BaTiO₃ FGM nanobeam for different n_x . $n_z = 0$, $F = -1$ nN, $V = -0.1$ V.

3.3 Combined influence of n_x and n_z

In this section the combined influence of the power-law exponents n_x and n_z is investigated by simultaneously change their values between 0 and 1. It has been demonstrated in the previous sections that values of the power-law exponents higher than 1 are monotone and almost dominated by the behavior of material 1. In Figure 13 the different previously proposed combinations of FGM are tested under different power-law exponent values. In Figures 13a and 13b the case of SrTiO₃-BaTiO₃ is treated for clamped-free and clamped-clamped configurations, respectively. In both cases, it appears that the maximum deflection is obtained when $n_x = 0$ and $n_z = 0$, meaning that the nanobeam is only by BaTiO₃. As for the FGM PbTiO₃-BaTiO₃, simulated in Figure 13c for the clamped-free configuration, the maximal amplitude is obtained for $n_x = 0$ and $n_z = 0.2$, indicating when the local effect of the effective flexoelectric bending moment is dissipated and replaced by a behavior dominated by the elasticity of the FGM. For the clamped-clamped configuration in Figure 13d, the maximum deflection is obtained for $n_x = 0.1$ and $n_z = 0$, demonstrating that the boundary condition changes the expected behavior because of the piezoelectric-flexoelectric effect difference between the two configurations. By introducing PVDF in the FGM combined with BaTiO₃, the response shown in Figures 13e and 13f demonstrate that a larger deflection is always obtained if the power-law exponents n_x and n_z are changed in the same time with the same value, for both clamped-free and clamped-clamped configurations.



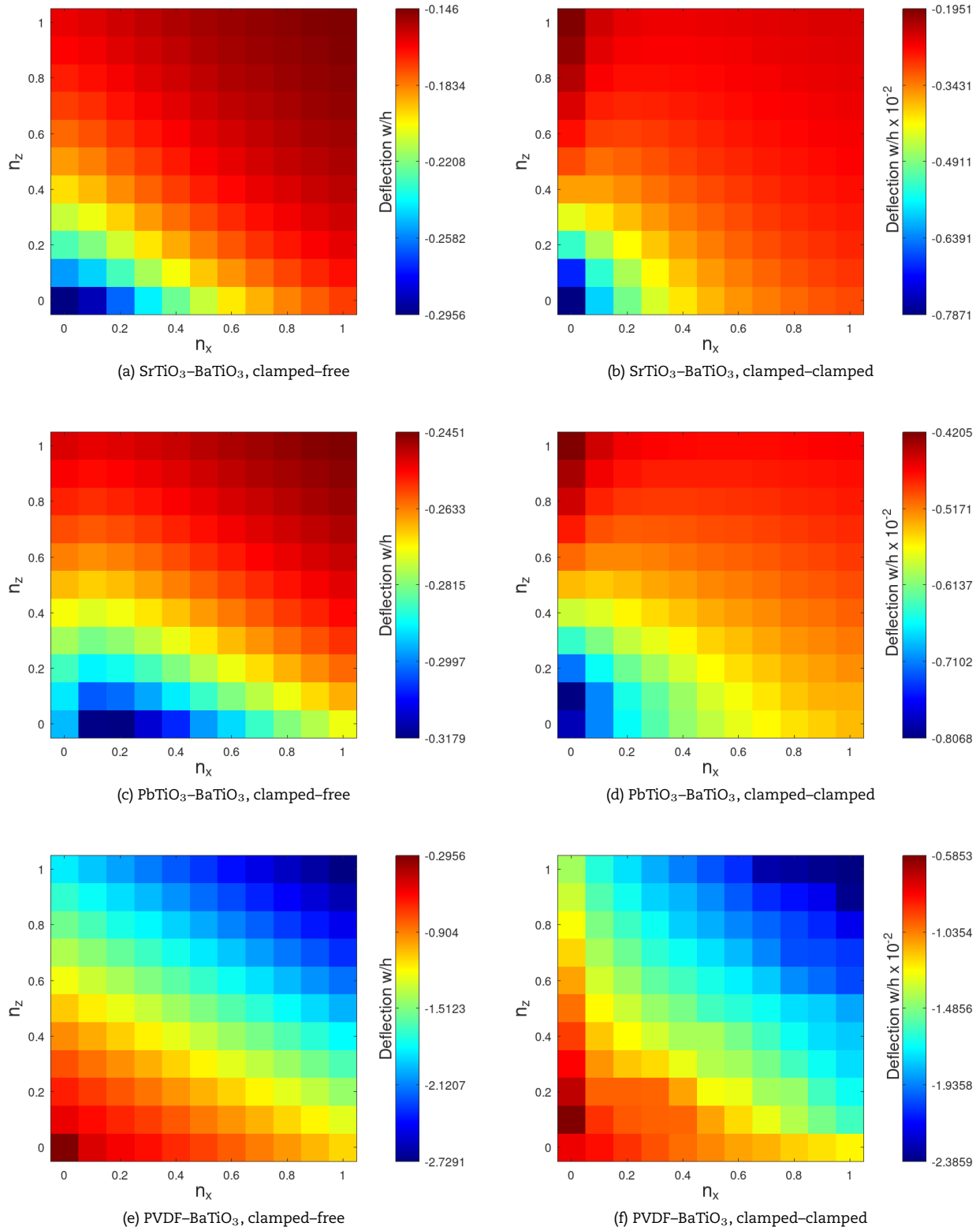


Fig. 13. Maximum deflections of FGM nanobeams for combined variation of n_x and n_z . $F = -1$ nN, $V = -0.1$ V.

4. Conclusion

In this study, a nonlinear finite element framework was developed for the static analysis of piezoelectric-flexoelectric nanobeams composed of functionally graded materials (FGMs). The model incorporates both piezoelectricity and flexoelectricity while accounting for material gradation in the longitudinal and transverse directions. By employing a mixed finite element formulation, Hermite shape functions, and the asymptotic numerical method, the model successfully captures nonlinear electromechanical coupling effects and size-dependent behavior.

Comparisons with previously reported homogeneous cases demonstrated that the proposed model achieves excellent agreement, thereby validating its accuracy. Beyond this validation, systematic analyses were carried out to examine the influence of the FGM gradation parameters n_x and n_z , as well as their combined effect for two set of boundary conditions, clamped-free and



clamped-clamped. The results confirmed that material distribution plays a decisive role in tuning the nanobeam's response. In particular, transverse gradation (n_z) was shown to significantly alter beam stiffness, while longitudinal gradation (n_x) introduced asymmetry in the deflection profile, especially in clamped-clamped configurations. These findings highlight the importance of controlling the spatial distribution of material phases to achieve tailored mechanical and electrical responses.

The parametric studies revealed that ceramic combinations with similar stiffness exhibited minor sensitivity to gradation, whereas hybrid combinations such as PVDF-BaTiO₃ displayed pronounced changes due to the large stiffness mismatch. Furthermore, the combined variation of n_x and n_z demonstrated non-monotonic trends, emphasizing the interplay between elasticity, piezoelectricity, and flexoelectricity in governing the electromechanical response. These insights are particularly relevant for the design of next-generation nanoelectromechanical systems, where optimizing performance requires precise control over material gradation and boundary conditions.

Overall, the present work provides a robust modeling tool and delivers new perspectives on the static behavior of FGM nanobeams. The outcomes are expected to support the design of advanced devices such as nanoscale sensors, actuators, and energy harvesters. Future investigations may extend this framework to dynamic analyses and complex geometries, while also exploring alternative gradation functions and topology optimization techniques to identify non-intuitive material distributions that maximize electromechanical efficiency in piezoelectric-flexoelectric nanosystems.

Author Contributions

I. Abed prepared the original draft, conducted the visualization, validation, software, and data curation; S. Abdelkhalek contributed to writing, review and editing, methodology, investigation, data curation, validation, and software; F. Najjar contributed to writing, methodology, investigation, conceptualization, project administration, funding acquisition, and supervision. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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