



Journal of Applied and Computational Mechanics



Research Paper

Vibration-based Energy Harvesting using a Van der Pol Oscillator under Harmonic Excitation

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Received June 28 2026; Revised September 28 2026; Accepted for publication September 28 2026.

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Abstract. This paper examines vibration-based electromagnetic energy harvesting (EH) using a driven van der Pol (vdP) oscillator with an external periodic excitation combined with a storage circuit. EH performance of the harvester system is investigated with the vdP oscillator and compared with that of a damped purely linear harmonic (DPLH) oscillator. Slow-flow equations describing vibration and output power solutions are established based on the averaging and harmonic balance methods. These methods yield approximate analytical solutions for periodic vibration and output power responses. The second-order perturbation method is applied to the slow-flow equations to obtain approximate quasiperiodic (QP) solutions for vibration and output power. The influence of the considered parameters on the output power is studied for periodic and QP regimes. This study demonstrates the advantage of using a vdP oscillator enabling EH in both the periodic and QP regimes. The results demonstrate that energy can be efficiently harvested in both regimes, depending on the selected system parameters. Numerical simulations were conducted to validate the analytical predictions.

Keywords: Energy harvesting; Van der Pol oscillator; Purely linear harmonic oscillator; Periodic; Quasiperiodic.

1. Introduction

The main purpose of designing vibration EH systems is to enhance the harvester's power generation under various environmental conditions. This is achieved by mechanical and electrical parameters that support strong oscillations over a wide range of operating conditions with low mechanical damping [1, 2]. Over the last few years, nonlinear vibration energy harvesting (EH) has become a fast-growing research field, driven by the need to address the narrow bandwidth of linear resonant harvesters and to exploit the rich nonlinear phenomena such as bistability, parametric resonance, internal resonances, external resonance [3], nonlinear tuned absorbers [4], self-excited oscillations [5, 6], auto-parametric systems [7], and cubic-quintic Duffing problems [8]. Other distinctive dynamical behaviors have also been reported, including bursting oscillations and various forms of bursting responses induced by pulse-shaped excitation [9-11]. Bursting oscillations are characterized by the alternation between small-amplitude quiescent states and large-amplitude oscillatory states. Such responses typically occur when the excitation frequency is much lower than the system's natural frequency, producing a slow-fast dynamical interaction. This mechanism can generate large-amplitude vibrations over a wider range of excitation conditions, thereby significantly broadening the effective operating bandwidth for EH. For instance, one way to improve EH performance is to exploit self-excited oscillations by using negative damping mechanisms. The vdP oscillator is a classical nonlinear system that has been extensively studied for its rich dynamical behavior such as limit cycles, QP oscillations, and synchronization phenomena under external forcing [12-14]. This oscillator is characterized by nonlinear damping that is negative at small amplitudes and positive at large amplitudes, enabling the generation of periodic self-sustained oscillations known as limit cycles. This behavior is due to the capacity of the system to inject energy at low amplitudes and dissipate it at high amplitudes, regulating the oscillation amplitude autonomously. In contrast to bistable or tristable systems, which depend on the potential energy well and require specific initial conditions or high enough excitation to sustain high-energy inter-well oscillations, the vdP oscillator can sustain self-excited limit-cycle oscillations without the need for any external energy input to sustain large-amplitude vibrations [15-17]. Synchronization properties of the forced vdP oscillator, including frequency locking and QP responses, have been extensively investigated [18, 19], indicating their potential for application in broadband EH. The QP responses or synchronization phenomena are generated by the interplay between the forcing frequency and the natural frequency limit cycle oscillator, which is due to the interplay of the external harmonic excitation and the self-sustained dynamics. This unique feature offers advantages like a strong amplitude response and less sensitivity to variations in parameters than conventional nonlinear harvesters.



The potential of self-excited oscillators for EH applications [20, 21] has been recently investigated. Galloping vibrations are self-excited vibrations [22, 23] that are caused by an aeroelastic instability in wind flows. For elastic bluff bodies, transverse galloping occurs at the critical wind speed, which leads to self-excited limit-cycle oscillations. Under the quasi-steady hypothesis [22], Barrero-Gil et al. [24] considered a one-degree-of-freedom (1-DOF) model. This model was used to investigate the mechanism of transverse galloping and the possibility of EH from fluid flows. Zhao et al. [25] studied the feasibility of various modeling approaches for 1-DOF galloping energy harvesters. Abdelkefi et al. [26] have worked on this with a 1-DOF system. The authors focused on the onset wind speed and EH performance of galloping-based piezoaeroelastic energy harvesters. Bibo and Daqaq [27] performed an analytical study of an energy harvester exposed to base excitations and wind-induced vibrations. Dai et al. [28] assessed the electromagnetic EH from galloping oscillations in 1-DOF systems; they showed the benefits of self-sustained oscillations. Likewise, Dash et al. [29] investigated a galloping-based EH. The results indicate that the power output is significantly reduced by forward inclinations, whereas backward inclinations exhibit a comparatively higher performance, providing insights into the enhancement of EH efficiency and structural reliability under extreme flow conditions. Wang et al. [30] considered a novel energy harvester and studied the transition of vortex-induced vibration (VIV) to galloping through numerical and experimental investigation. The findings indicate that the Y-shaped design can effectively improve EH performance, expand the operating wind speed range, and outperform conventional VIV and square-section galloping harvesters. Recently, El Moussati et al. [31] investigated an electromagnetic energy harvester exploiting the combined effects of these two aerodynamic phenomena in a 2-DOF system. The energy harvesting performance of a galloping-only configuration is compared with that of a coupled galloping-VIV configuration to assess the benefits of multi-instability excitation, particularly at low wind speeds. Time-delay control has been shown to be a powerful tool to enhance EH in nonlinear oscillators. Bichri et al. studied a delay-controlled Rayleigh-Duffing oscillator connected to an energy harvester subsystem [32]. The results revealed that time-delay control can greatly improve QP EH, especially under off-resonance operating conditions. Kirrou et al. [28] investigate a delayed Rayleigh-Duffing piezoelectric energy harvester near 3-subharmonic resonance. The results show that time-delay control is effective in improving the QP EH and expanding the bandwidth of high-power operation. Moreover, coupling different vibration mechanisms has recently received considerable attention in the scope of EH applications [34, 35]. In the context of electromagnetic harvesters, the mutual influence of self-excitation and parametric resonance has been studied for self-sustained oscillations.

Despite these advances, the direct application of the vdP oscillator mechanism to vibration EH under harmonic base excitation has received limited systematic investigation. The inherent ability of the vdP oscillator to self-regulate its amplitude through nonlinear damping, combined with its entrainment capabilities when externally driven [36, 37], offers a potentially robust harvesting mechanism that may overcome some limitations of conventional nonlinear harvesters. Most of the literature has focused on the best mechanical model for output performance of harvester systems. From the electrical circuit point of view, in the most studied case is when the harvester is connected to classical linear circuits, scientists have coupled different transduction mechanisms [38] to the mechanical part. The situation in which the mechanical subsystem is coupled to a nonlinear self-excited circuit has been relatively less studied. Recently, El Moussati et al. [39, 40] compared a conventional RLC circuit interface with a circuit incorporating the van der Pol model. In [39], the mechanical system is subjected to galloping vibrations, while in [40], it is subjected to periodic excitation. They have shown that the vdP circuit can performed of the scavenged energy. Also, considerable advances have been reported in VIV-based EH; the effect of a vdP circuit on EH performance is still largely unknown, and previous studies have been limited to classical resistive circuits [41-44]. In this work, we investigate the EH potential of a vdP oscillator under harmonic excitation, using averaging and harmonic balance methods for periodic responses, and, significantly, apply a second-step perturbation technique to reveal novel mechanisms of EH within the QP regime. The EH is tested under different operating conditions and parameter settings. The results indicate that the self-sustained vdP oscillator can provide stable large-amplitude oscillations and output power for a large range of parameters, which is a significant advantage over DPLH harvesters in terms of efficiency and performance. In fact, QP vibrations can emerge when the external excitation frequency interacts with the natural frequency of the vdP oscillator. More importantly, the QP regime can greatly enhance EH performance owing to the sustained large vibration amplitude over a wide frequency range and thus the increased harvested power. These results demonstrate that the QP regime can be used as an efficient mechanism to improve the robustness and efficiency of EH for different excitation conditions.

In the rest of the manuscript, Section 2 provides a description of the proposed EH system, which consists of a harmonically forced vdP oscillator coupled to a linear storage circuit. Approximations of the periodic responses of vibration and output power are derived based on averaging and harmonic balance methods. Section 3 presents the main results concerning periodic vibration and output power responses. The effects of the various parameters on the system responses are investigated. The study demonstrates that the self-sustained nature of the vdP oscillator can provide stable high-amplitude oscillations over a broad frequency range, offering advantages over DPLH harvesters in EH performance. In Section 4, the QP responses for vibration and output power are approximated using a second-step perturbation technique. Then the performance of the EH in the QP regime is investigated. Section 5 presents the conclusions of the paper.

2. Description and Modeling of the Energy Harvester

The harvester system under consideration consists of two coupled components: a mechanical subsystem modeled as a vdP-type oscillator subjected to a harmonic excitation $F(t) = f \cos(\Omega t)$, and an electrical subsystem represented by a linear RL storage circuit. The electromechanical coupling between the two subsystems arises from the electromagnetic interaction between a permanent magnet and a coil, in accordance with Faraday's induction law. The proposed model is sketched in Fig. 1.

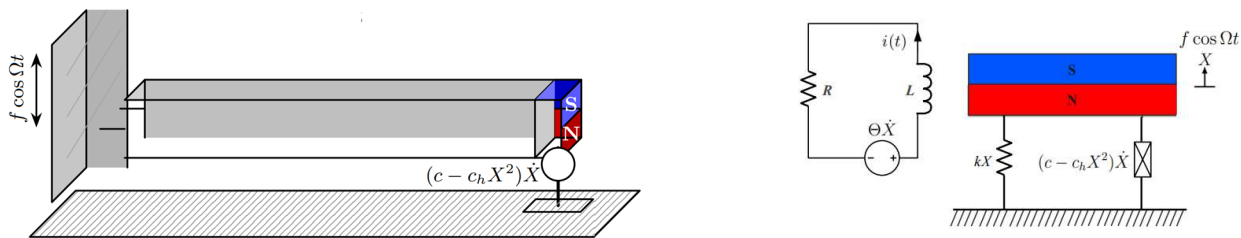


Fig. 1. Sketch of the proposed harvester system; Schematic configuration of the structure (left panel), Storage RL interface circuit (middle panel), vdP oscillator (right panel), (Color online).



The equations governing the coupled electromechanical dynamics take the following form:

$$m\ddot{X} - (c - c_h X^2)\dot{X} + kX - \Theta I = f \cos \Omega t \quad (1)$$

$$L \frac{dI}{dt} + RI = -\Theta \dot{X} \quad (2)$$

where f and Ω denote the amplitude and angular frequency of the external harmonic excitation, respectively; Θ is the electromechanical coupling parameter; L is the coil inductance; R is the load; k is the linear stiffness parameter; c and c_h are the linear and nonlinear damping coefficients characterizing the vdP term; X is the mechanical structure displacement; and I is the output current flowing through the circuit.

Introducing the dimensionless variables $x = X/x_0$ and $i = Li/(\Theta x_0)$, the dimensional equations (1)–(2) are transformed into the following nondimensional form:

$$\ddot{x} - (\alpha - \beta x^2)\dot{x} + \omega_0^2 x - \chi i = F \cos \omega \tau \quad (3)$$

$$\frac{di}{d\tau} + \lambda i = -\dot{x} \quad (4)$$

where the dimensionless parameters are defined as $\omega = \Omega/\omega_0$, $\tau = \omega_0 t$, $\omega_0 = (k/m)^{0.5}$, $\chi = \Theta^2/(Lm\omega_0^2)$, $\lambda = 1/(L\omega_0)$, $\alpha = c/(m\omega_0)$, $\beta = c_h x_0^2/(m\omega_0)$, $F = f/(mx_0\omega_0^2)$.

The analysis is performed in the neighborhood of the principal resonance defined by:

$$\omega_0^2 = \omega^2 + \varepsilon \sigma \quad (5)$$

in which σ is a small detuning parameter frequency and ε is a small positive bookkeeping parameter. The slow-flow equations of amplitude and phase describing the system response near resonance are obtained by the averaging method and harmonic balance technique [12, 13].

To this end, system (3)–(4) is reformulated as:

$$\ddot{x} + \omega^2 x = \varepsilon G(x, \dot{x}, i, \tau) \quad (6)$$

$$\frac{di}{d\tau} + \lambda i = -\dot{x} \quad (7)$$

where $G(x, \dot{x}, i, \tau) = -\sigma x + (\alpha - \beta x^2)\dot{x} + \chi i + F \cos \omega \tau$, with the rescaling $\chi = \varepsilon \chi$, $F = \varepsilon F$, $\alpha = \varepsilon \alpha$, and $\beta = \varepsilon \beta$.

The solutions to Eqs. (6)–(7) are expressed as:

$$x(\tau) = R \cos(\omega \tau - \varphi_1) \quad (8a)$$

$$i(\tau) = I \cos(\omega \tau - \varphi_2) \quad (8b)$$

where the amplitudes R and I , and the phases φ_1 and φ_2 , vary slowly and can be considered approximately constant over one oscillation period $2\pi/\omega$. They are determined by:

$$\dot{R} = -\frac{\varepsilon}{2\pi} \int_0^{2\pi} \sin(\omega \tau - \varphi_1) G(x, \dot{x}, i, \tau) d\tau \quad (9a)$$

$$R\dot{\varphi}_1 = \frac{\varepsilon}{2\pi} \int_0^{2\pi} \cos(\omega \tau - \varphi_1) G(x, \dot{x}, i, \tau) d\tau \quad (9b)$$

$$\dot{I} + \dot{R} \cos(\varphi_1 - \varphi_2) - R\dot{\varphi}_1 \sin(\varphi_1 - \varphi_2) = -\lambda I + \omega R \sin(\varphi_1 - \varphi_2) \quad (9c)$$

$$I\dot{\varphi}_2 + \dot{R} \sin(\varphi_1 - \varphi_2) + R\dot{\varphi}_1 \cos(\varphi_1 - \varphi_2) = -\omega I - \omega R \cos(\varphi_1 - \varphi_2) \quad (9d)$$

By performing the integration, the modulation equations are obtained as:

$$\dot{R} = k_1 R + k_2 R^3 + k_3 \sin \varphi_1 + \frac{\chi}{\omega} I \sin(\varphi_1 - \varphi_2) \quad (10a)$$

$$R\dot{\varphi}_1 = k_4 R + k_3 \cos \varphi_1 + \frac{\chi}{\omega} I \cos(\varphi_1 - \varphi_2) \quad (10b)$$

$$\dot{I} + \dot{R} \cos(\varphi_1 - \varphi_2) - R\dot{\varphi}_1 \sin(\varphi_1 - \varphi_2) = -\lambda I + \omega R \sin(\varphi_1 - \varphi_2) \quad (10c)$$

$$I\dot{\varphi}_2 + \dot{R} \sin(\varphi_1 - \varphi_2) + R\dot{\varphi}_1 \cos(\varphi_1 - \varphi_2) = -\omega I - \omega R \cos(\varphi_1 - \varphi_2) \quad (10d)$$

where the differentiation is taken according to the new time $T = \tau/\varepsilon$, $k_1 = \alpha/2$, $k_2 = -\beta/8$, $k_3 = F/(2\omega)$ and $k_4 = -\sigma/(2\omega)$.

Fixed-point equilibria of the modulation equations (10), corresponding to periodic responses of the system equations (3)–(4), are obtained by imposing the stationarity conditions $\dot{R} = \dot{I} = \dot{\varphi}_1 = \dot{\varphi}_2 = 0$. Eliminating the phase angles φ_1 and φ_2 by substituting the expressions for $\cos(\varphi_1 - \varphi_2)$ and $\sin(\varphi_1 - \varphi_2)$ derived from Eqs. (10c)–(10d) into Eqs. (10a)–(10b), then squaring and summing the resulting equations while invoking the Pythagorean identity, one obtains the following algebraic system in terms of $\rho = R^2$ and I :

$$\rho \left(k_1 + \frac{\chi \lambda}{\lambda^2 + \omega^2} + k_2 \rho \right)^2 + \rho \left(k_4 - \frac{\chi \omega}{\lambda^2 + \omega^2} \right)^2 = k_3^2 \quad (11a)$$



$$I = \frac{\omega}{\sqrt{\lambda^2 + \omega^2}} R \quad (11b)$$

$$\sin(\varphi_1 - \varphi_2) = \frac{\lambda}{\sqrt{\lambda^2 + \omega^2}} \quad (11c)$$

$$\cos(\varphi_1 - \varphi_2) = -\frac{\omega}{\sqrt{\lambda^2 + \omega^2}} \quad (11d)$$

The instantaneous power $P(\tau)$ defined as:

$$P(\tau) = \lambda i(\tau)^2 \quad (12)$$

Hence, the average generation power P_{av} is obtained by averaging the instantaneous power $P(t)$ given by Eq. (12) over one modulation period, which yields:

$$P_{av} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} P(\tau) d\tau \quad (13)$$

Using Eqs. (8b), (11b), and (12), the maximum generation power $P_{\max}$ is derived as:

$$P_{\max} = \frac{\lambda\omega^2}{\lambda^2 + \omega^2} R^2 \quad (14)$$

3. Periodic Vibration and Energy Harvesting

In this section, Eq. (11a) is solved by using Cardano's method [5], and we fix the parameters $\alpha = 0.1$ ($\alpha = -0.1$), $\beta = 0.1$ ($\beta = 0$) for the (without) vdP oscillator. The circles inset in the figures correspond to the numerical integration results of Eqs. (3)–(4) obtained by means of the fourth-order Runge-Kutta (RK4) method [6], showing good agreement with the analytical predictions. Also, the findings obtained with a vdP oscillator are compared with those of a DPLH oscillator. The dashed grey lines are for the case with a DPLH oscillator, and the black lines are for the case with a vdP oscillator. Readers are referred to the color version of this paper in the web version.

Next, enhanced and sustained performance of EH is investigated in the periodic response for the selected parameter values. Figure 2 illustrates the frequency-response curves of vibration (Fig. 2(a)) and output power (Fig. 2(b)), obtained by solving Eqs. (11a)–(11b) and (14) using Cardano's method, for a vdP oscillator (black line) and for only a DPLH oscillator (dashed grey line) for $\chi = 1$, $\lambda = 1$ and $F = 1$. Circles represent numerical simulation results obtained via the RK4 method. For the selected parameter values, the frequency responses present the resonance peak. As illustrated in the figure, coupling the harvester with a vdP oscillator significantly increases both vibration and extracted power amplitude s around resonance compared with the configuration employing only the DPLH oscillator. Particularly, as presented in Fig. 2(b), the peak output power is increased by approximately 57.6% when the vdP oscillator is incorporated. This enhancement is mainly due to the self-excitation effect introduced by the vdP nonlinearity, which effectively boosts the system response near resonance.

Figure 3 shows the corresponding time histories obtained via the RK4 method of the vibration $x(t)$ and the harvested power $P(t)$ picked from Fig. 2 at $\omega = 1.27$, showing good agreement with the analytical predictions obtained in Fig. 2.

Figure 4 illustrates the periodic vibration and output power amplitudes, obtained by using Eqs. (11a)–(11b) and (14) using Cardano's method, as a function F for $\chi = 1$, $\lambda = 1$ and $\omega = 1.27$. Circles represent numerical simulation results obtained via the RK4 method. The results for the harvester system with a vdP oscillator (black line) are compared with those of the DPLH (dashed grey line) oscillator. From Fig. 4(b), it can be seen that the vdP oscillator significantly improves the output power in a wide range of excitation amplitudes F .

Figure 5 displays the periodic solutions of vibration and output power, obtained by solving Eqs. (11a)–(11b) and (14) using Cardano's method, as functions of the dimensionless load parameter λ for $\chi = 1$, $F = 1$ and $\omega = 1.27$. Circles represent numerical simulation results obtained via the RK4 method. The figure demonstrates the superior EH of the harvester system using a vdP oscillator compared with using the DPLH oscillator. It can be observed that the vdP oscillator-based system consistently yields higher generation power amplitudes over the entire range of λ . Specifically, the harvested power exhibits better performance, with an approximate improvement of 49%, as illustrated in Fig. 5(b) for $\lambda = 1$.

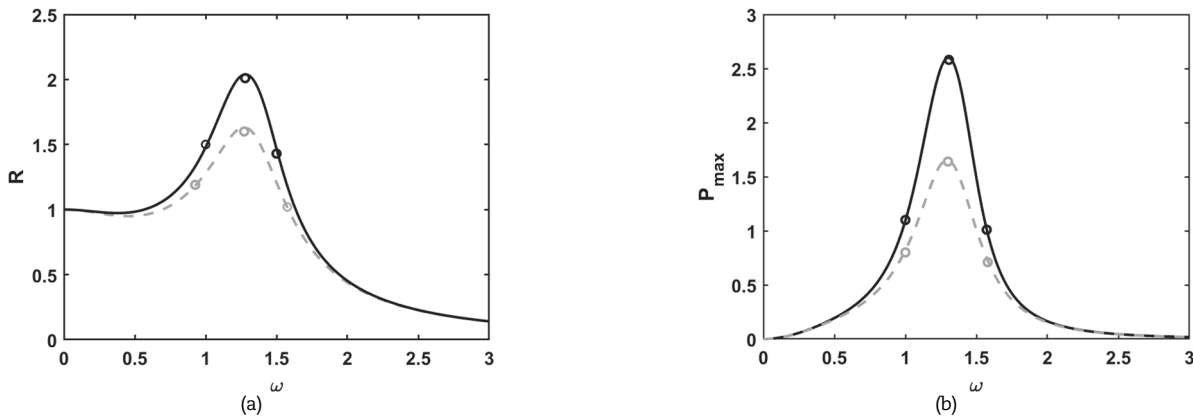


Fig. 2. Frequency-response curves obtained by using Cardano's method for $\chi = 1$, $\lambda = 1$ and $F = 1$; (a) Vibration amplitude, (b) Generation power amplitude; Black (dashed grey) line with (without) vdP oscillator; Circles for numerical simulations obtained via the RK4 method.



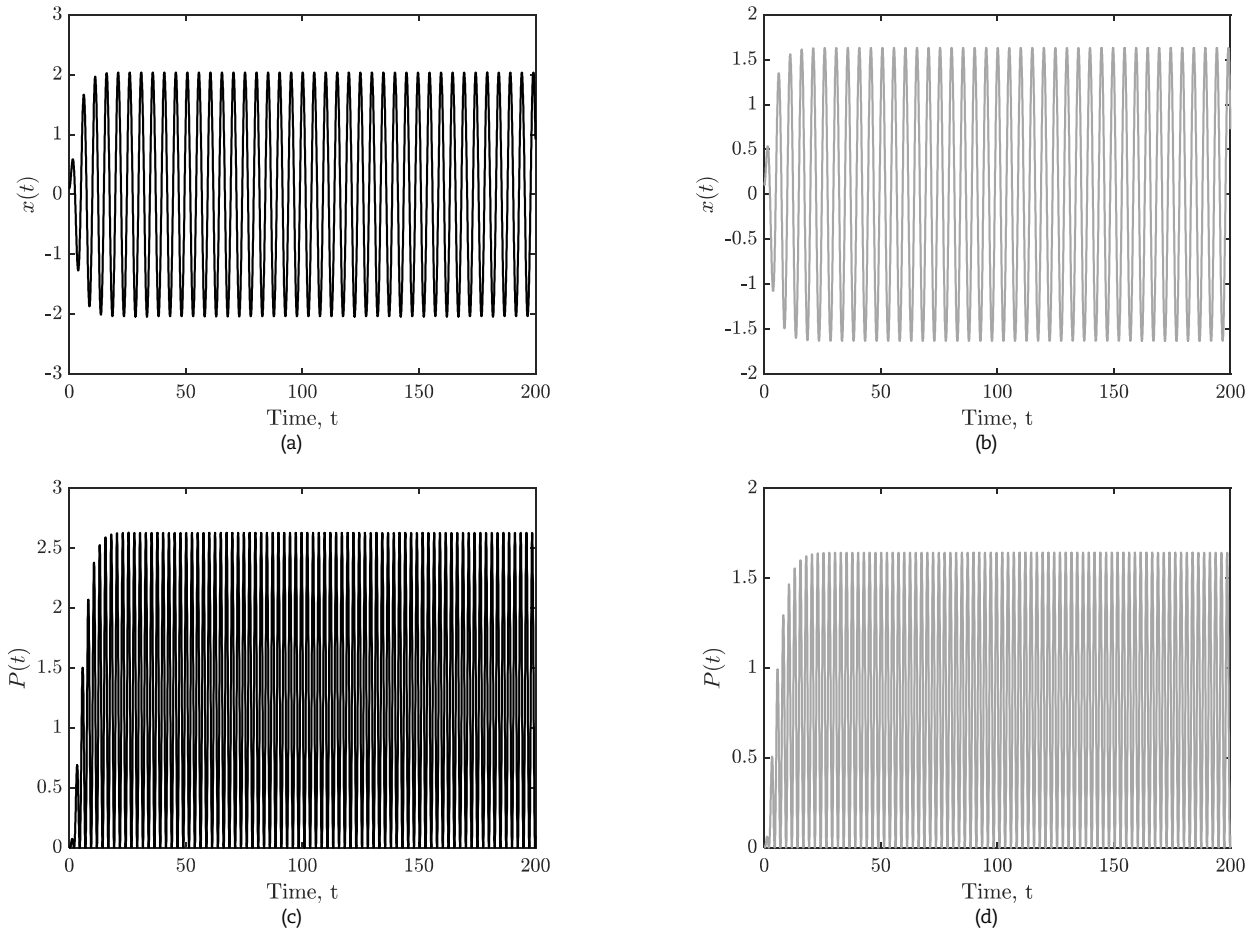


Fig. 3. Time histories picked from Fig. 2 for $\omega = 1.27$ obtained via numerical integration of the governing equations using the RK4 method; (a), (b) Vibration displacement $x(t)$ and (c), (d) output power $P(\tau)$; Black lines correspond to the system with the vdP oscillator ((a) and (c)), and grey lines correspond to the system without the vdP oscillator ((b) and (d)).

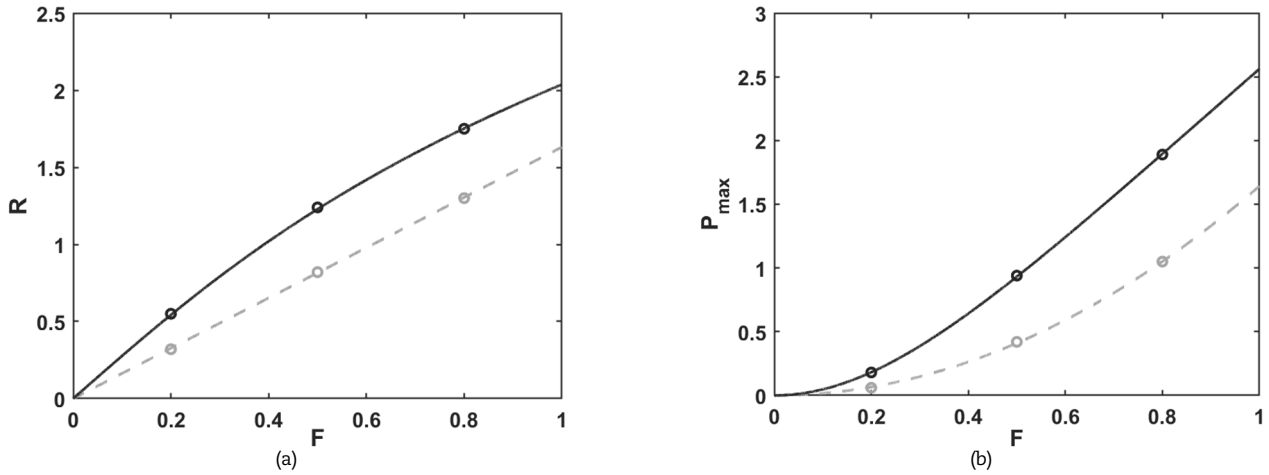


Fig. 4. Amplitude responses versus forcing amplitude F obtained by using Cardano's method for $\chi = 1$, $\lambda = 1$ and $\omega = 1.27$; (a) Vibration, (b) Output power; Black (dashed grey) line for a vdP (DPLH) oscillator; Circles for numerical simulations obtained via the RK4 method.

Figure 6 shows the vibration amplitude and the corresponding generation power, obtained by solving Eqs. (11a)–(11b) and (14) using Cardano's method, as a function χ for $F = 1$, $\omega = 1.27$ and $\lambda = 1$. Circles represent numerical simulation results obtained via the RK4 method. This figure highlights EH with the vdP oscillator compared to the DPLH harvester. Namely, as illustrated in Fig. 6(b), the harvested output power is improved at the optimal value $\chi = 0.572$.

4. Quasiperiodic Vibration and Energy Harvesting

Throughout this section, the linear and nonlinear damping coefficients characterizing the vdP term are set to $\alpha = 0.8$ and $\beta = 0.01$. For these choices, the periodic response loses stability over a large range of the considered parameters, leading to QP solutions. The subsequent analysis demonstrates that robust and sustained EH performance is achievable in the QP regime across wide parameter intervals, extending also to regions away from resonance.



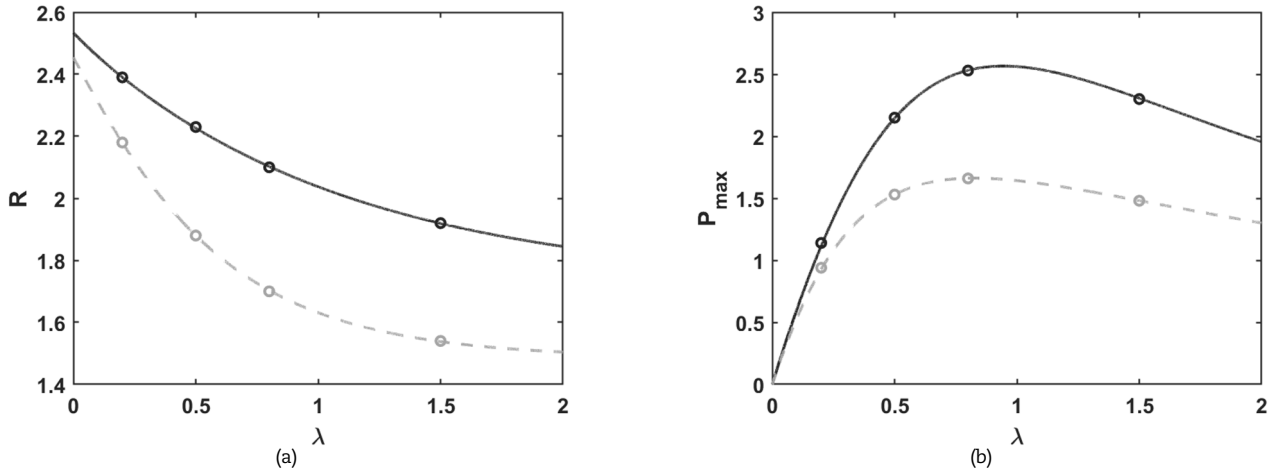


Fig. 5. Amplitude responses versus the load parameter λ obtained by using Cardano's method for $\chi = 1$, $F = 1$ and $\omega = 1.27$; (a) Vibration, (b) Output power; Black line for a vdP oscillator and dashed grey line for a DPLH oscillator; Circles for numerical simulations obtained via the RK4 method.

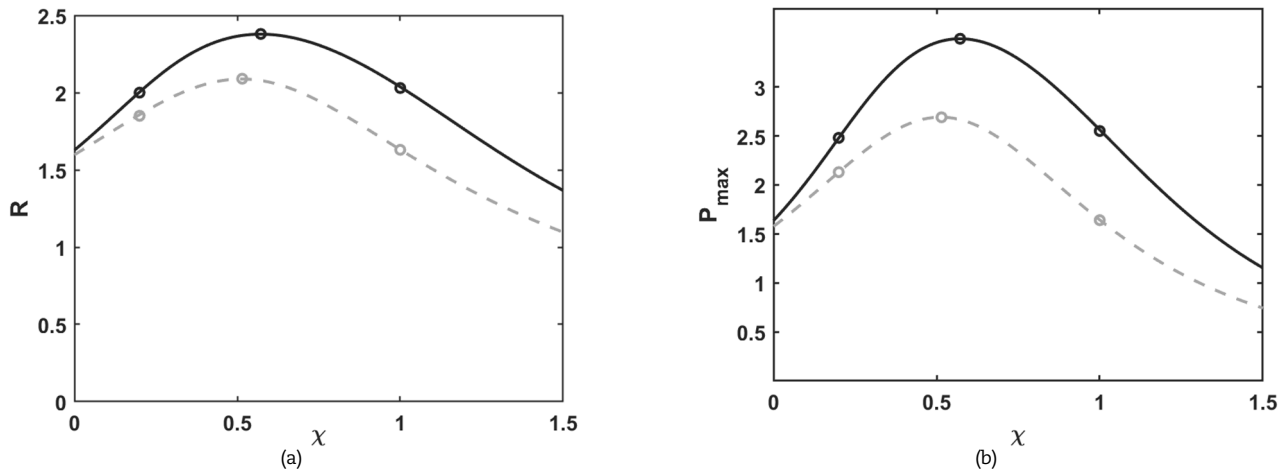


Fig. 6. Amplitude responses versus the electromechanical coupling coefficient χ obtained by using Cardano's method for $F = 1$, $\omega = 1.27$ and $\lambda = 1$; (a) Vibration, (b) Output power; Black (dashed grey) line for a vdP (DPLH) oscillator; Circles for numerical simulations obtained via the RK4 method.

When the vdP oscillator is embedded in the harvester system, if the limit-cycle oscillation frequency is not synchronized with the external excitation frequency, leading to QP vibrations when there is no frequency locking (synchronization). In this section, we focus on an analytical approximation for the QP solutions for the response amplitudes.

To construct approximate QP solutions, we employ the second-step perturbation method [33]. The slow-flow system (10) possesses an invariance symmetry under the simultaneous transformations $\varphi_1 \rightarrow -\varphi_1$, $\varphi_2 \rightarrow -\varphi_2$, $\omega \rightarrow -\omega$, $k_3 \rightarrow -k_3$, and $k_4 \rightarrow -k_4$. Exploiting this symmetry, system (10) is recast in the equivalent form:

$$\dot{R} = k_1 R + k_2 R^3 + s k_3 \sin \varphi_1 + \frac{\chi}{s \omega} I \sin(\varphi_1 - \varphi_2) \quad (15a)$$

$$R \dot{\varphi}_1 = s k_4 R + s k_3 \cos \varphi_1 + \frac{\chi}{s \omega} I \cos(\varphi_1 - \varphi_2) \quad (15b)$$

$$\dot{I} + \dot{R} \cos(\varphi_1 - \varphi_2) - R \dot{\varphi}_1 \sin(\varphi_1 - \varphi_2) = -\lambda I + s \omega R \sin(\varphi_1 - \varphi_2) \quad (15c)$$

$$I \dot{\varphi}_2 + \dot{R} \sin(\varphi_1 - \varphi_2) + R \dot{\varphi}_1 \cos(\varphi_1 - \varphi_2) = -s \omega I - s \omega R \cos(\varphi_1 - \varphi_2) \quad (15d)$$

where $s = \pm 1$. This symmetry will be used to construct the modulation envelope of the QP response in both the vibration and the generation power.

To proceed with the second-step perturbation analysis [45], the polar slow-flow equations (15) are converted to Cartesian coordinates via the substitutions $u = R \cos \varphi_1$, $v = -R \sin \varphi_1$, $y = I \cos \varphi_2$, and $z = -I \sin \varphi_2$, yielding: $\frac{du}{dT} = \dot{R} \cos \varphi_1 - R \dot{\varphi}_1 \sin \varphi_1$, $\frac{dv}{dT} = -\dot{R} \sin \varphi_1 - R \dot{\varphi}_1 \cos \varphi_1$, $\frac{dy}{dT} = \dot{I} \cos \varphi_2 - I \dot{\varphi}_2 \sin \varphi_2$, $\frac{dz}{dT} = -\dot{I} \sin \varphi_2 - I \dot{\varphi}_2 \cos \varphi_2$, which leads to:

$$\frac{du}{dT} = s k_4 v + \eta \left\{ k_1 u + k_2 u(u^2 + v^2) + \frac{\chi}{s \omega} z \right\} \quad (16a)$$

$$\frac{dv}{dT} = -s k_4 u - s k_3 + \eta \left\{ k_1 v + k_2 v(u^2 + v^2) - \frac{\chi}{s \omega} y \right\} \quad (16b)$$

$$\frac{dy}{dT} = -\lambda y - \frac{du}{dT} - \eta \{ s \omega v + s \omega z \} \quad (16c)$$



$$\frac{dz}{dT} = -\lambda z - \frac{dv}{dT} + \eta\{s\omega u + s\omega y\} \quad (16d)$$

where η is a bookkeeping parameter introduced to scale the damping, nonlinear, and electromechanical coupling terms. Periodic solutions of the Cartesian slow-flow (16) correspond to QP solutions of the system (3)–(4).

Approximate periodic solutions of Eq. (16) are sought via the multiple scales method [12], with expansions of the form:

$$u(T_0, T_1) = u_0(T_0, T_1) + \eta u_1(T_0, T_1) + O(\eta^2) \quad (17a)$$

$$v(T_0, T_1) = v_0(T_0, T_1) + \eta v_1(T_0, T_1) + O(\eta^2) \quad (17b)$$

$$y(T_0, T_1) = y_0(T_0, T_1) + \eta y_1(T_0, T_1) + O(\eta^2) \quad (17c)$$

$$z(T_0, T_1) = z_0(T_0, T_1) + \eta z_1(T_0, T_1) + O(\eta^2) \quad (17d)$$

with $T_0 = T$ and $T_1 = \eta T$. Defining $D_i = \partial/\partial T_i$ so that $d/dT = D_0 + \eta D_1 + O(\eta^2)$ and then $\frac{d^2}{dT^2} = [D_0 + \eta D_1]^2 = D_0^2 + 2\eta D_0 D_1 + \eta^2 D_1^2 + O(\eta^2)$. Substituting Eq. (17) into Eq. (16), one obtains:

$$\begin{aligned} [D_0 + \eta D_1]^2(u_0 + \eta u_1) &= -k_4^2(u_0 + \eta u_1) - k_3 k_4 + \eta s k_4 \{k_1(v_0 + \eta v_1) - k_2(v_0 + \eta v_1)((u_0 + \eta u_1)^2 + (v_0 + \eta v_1)^2) - \frac{\chi}{s\omega}(y_0 + \eta y_1)\} \\ &+ \eta [D_0 + \eta D_1] \{k_1(u_0 + \eta u_1) + k_2(u_0 + \eta u_1)((u_0 + \eta u_1)^2 + (v_0 + \eta v_1)^2) + \frac{\chi}{s\omega}(z_0 + \eta z_1)\} \end{aligned} \quad (18a)$$

$$[D_0 + \eta D_1](u_0 + \eta u_1) = s k_4(v_0 + \eta v_1) + \eta \left\{ k_1(u_0 + \eta u_1) + k_2(u_0 + \eta u_1)((u_0 + \eta u_1)^2 + (v_0 + \eta v_1)^2) + \frac{\chi}{s\omega}(z_0 + \eta z_1) \right\} \quad (18b)$$

$$[D_0 + \eta D_1] = -\lambda(y_0 + \eta y_1) - [D_0 + \eta D_1](u_0 + \eta u_1) - \eta \{s\omega(v_0 + \eta v_1) + s\omega(z_0 + \eta z_1)\} \quad (18c)$$

$$[D_0 + \eta D_1](z_0 + \eta z_1) = -\lambda(z_0 + \eta z_1) - [D_0 + \eta D_1](v_0 + \eta v_1) + \eta \{s\omega(u_0 + \eta u_1) + s\omega(y_0 + \eta y_1)\} \quad (18d)$$

and then collecting terms at successive orders of η , one gets up to the first and second order:

$$D_0^2 u_0 + \nu^2 u_0 = s^2 k_3 k_4 \quad (19a)$$

$$s k_4 v_0 = D_0 u_0 \quad (19b)$$

$$D_0 y_0 + \lambda y_0 = -D_0 u_0 \quad (19c)$$

$$D_0 z_0 + \lambda z_0 = -D_0 v_0 \quad (19d)$$

and

$$D_0^2 u_1 + \nu^2 u_1 = s k_4 \left\{ -D_1 v_0 + k_1 v_0 + k_2 v_0(u_0^2 + v_0^2) - \frac{\chi}{s\omega} y_0 \right\} - 2D_0 D_1 u_0 + D_0 \left[k_1 u_0 + k_2 u_0(u_0^2 + v_0^2) + \frac{\chi}{s\omega} z_0 \right] \quad (20a)$$

$$s k_4 v_1 = D_0 u_1 + D_1 u_0 - k_1 u_0 - k_2 u_0(u_0^2 + v_0^2) - \frac{\chi}{s\omega} z_0 \quad (20b)$$

$$D_0 y_1 + \lambda y_1 = -D_1 y_0 - (D_0 u_1 + D_1 u_0) - s\omega v_0 - s\omega z_0 \quad (20c)$$

$$D_0 z_1 + \lambda z_1 = -D_1 z_0 - (D_0 v_1 + D_1 v_0) - s\omega u_0 + s\omega y_0 \quad (20d)$$

Here, $\nu = s k_4$ is the frequency of the periodic orbit in the slow-flow system, which corresponds to the modulation frequency of the QP oscillation in the original system.

The first-order solutions are given by:

$$u_0(T_0, T_1) = r(T_1) \cos(\nu T_0 + \theta(T_1)) - \frac{k_3}{k_4} \quad (21a)$$

$$v_0(T_0, T_1) = -r(T_1) \sin(\nu T_0 + \theta(T_1)) \quad (21b)$$

$$y_0(T_0, T_1) = M r(T_1) [\lambda \sin(\nu T_0 + \theta(T_1)) - \nu \cos(\nu T_0 + \theta(T_1))] \quad (21c)$$

$$z_0(T_0, T_1) = M r(T_1) [\lambda \cos(\nu T_0 + \theta(T_1)) + \nu \sin(\nu T_0 + \theta(T_1))] \quad (21d)$$

where $M = \nu/(\lambda^2 + \nu^2)$, and $r(T_1)$ and $\theta(T_1)$ denote the amplitude and phase, respectively, of the slow-flow limit cycle. Inserting Eq. (21) into the first-order equations (20a)–(20d) and eliminating the secular terms generated in Eq. (20a) yields the following autonomous slow-slow flow equations:

$$\frac{dr}{dT_1} = \left(k_1 + \frac{M\lambda\chi}{s\omega} + \frac{2k_2 k_3^2}{k_4^2} \right) r + k_2 r^3 \quad (22a)$$



$$r \frac{d\theta}{dT_1} = \frac{S\chi k_4}{\omega} r \quad (22b)$$

Nontrivial steady-state solutions of (22), obtained by setting $dr/dT_1 = 0$, yield the QP vibration amplitude:

$$r = \sqrt{-\left(k_1 + \frac{M\lambda\chi}{s\omega} + \frac{2k_2k_3^2}{k_4^2}\right)/k_2} \quad (23)$$

The corresponding current amplitude $I = (y_0^2 + z_0^2)^{0.5}$ is given by:

$$I = M r \sqrt{\lambda^2 + \nu^2} = \frac{\nu}{\sqrt{\lambda^2 + \nu^2}} r \quad (24)$$

The nontrivial fixed point in Eq. (22) defines the amplitude of the limit cycle in the slow-flow system, and hence governs the envelope of the QP vibration in the original harvester system (3)–(4). The modulated amplitude of the QP response $R = (u_0^2 + v_0^2)^{0.5}$, evaluated from Eq. (21), takes the explicit form:

$$R_{\max,\min} = \max, \min \left\{ \sqrt{r^2 - 2r \frac{k_3}{k_4} \cos\left(\nu T_0 + \theta(T_1)\right) + \frac{k_3^2}{k_4^2}} \right\} \quad (25)$$

Following the approach in [22], the maximum output power in the QP modulation regime is expressed as:

$$P_{\max QP} = \frac{\lambda \nu^2}{\lambda^2 + \nu^2} r^2 \quad (26)$$

Figure 7 presents the frequency-response curves for the periodic (blue lines) and QP (green lines) vibration and output power amplitudes. These curves are obtained by solving Eqs. (11a)–(11b) and (14) for the periodic case, and Eqs. (23)–(26) for the QP case, with a vdP oscillator and fixed parameters $F = 0.7$, $\lambda = 0.5$, and $\chi = 0.2$. The connected green circles are the numerical simulation results obtained via the RK4 method. The results reveal that the periodic solution remains unstable throughout most of the frequency domain (shown as dotted black lines), while stable QP vibrations (solid green lines) emerge over a wide frequency band wherever periodic solutions lose stability. Figure 7(b) further confirms that the QP regime enables sustained energy harvesting across this broad frequency interval. These findings underscore the advantage of the vdP oscillator in enabling broadband electromagnetic energy harvesting through QP dynamics. Time histories obtained by using the RK4 method of the vibration displacement $x(t)$ and generation power $P(t)$ in the QP regime are displayed in Figs. 8(a) and 8(b), respectively, extracted from Fig. 7 at $\omega = 1.5$, as indicated by the connected green circles in the inset.

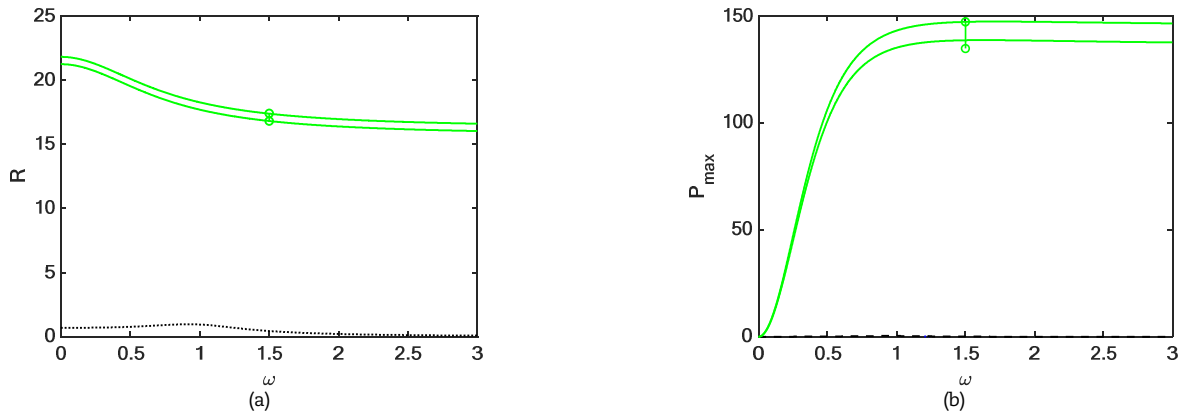


Fig. 7. Amplitude responses versus frequency ω with a vdP oscillator for $F = 0.7$, $\lambda = 0.5$ and $\chi = 0.2$, (a) Vibration, (b) Output power; Dotted black lines for unstable periodic solutions; Green lines for analytical responses of QP modulation. Connected green circles for numerical simulations obtained via the RK4 method (Color online).

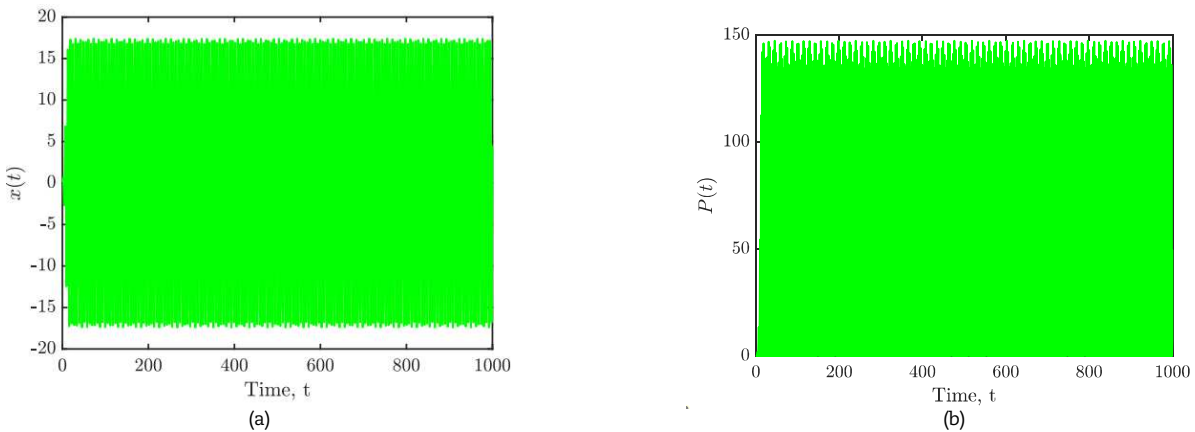


Fig. 8. Time histories of vibration $x(t)$ and (a) Output power $P(t)$, (b) Selected from Fig. 7 for $\omega = 1.5$ obtained by numerical integration of the governing equations with a vdP oscillator using the RK4 method (Color online).



Figures 9(a) and 9(b) show the amplitudes of the vibration and the generation power versus F , which display results for both the periodic (black lines) and QP (green lines) regimes, computed from Eqs. (11a)–(11b), (14), and (23)–(26). The connected green circles are the numerical simulation results obtained via the RK4 method. The stable QP branch (solid green lines) is present for small values of F and persists up to $F \approx 2.7$, beyond which the stable periodic branch (solid black line) emerges and grows monotonically. The unstable periodic branch (dotted black line) exhibits the characteristic folded topology associated with a saddle-node bifurcation. As shown in Fig. 9(b), the QP regime constitutes the sole source of harvestable energy at low excitation levels, while at higher F values, the periodic regime contributes to EH. For low excitation amplitude F , the external forcing is insufficient to synchronize with the self-sustained oscillations of the vdP mechanism, leading to QP response, which serves as the primary source of harvestable energy. Conversely, higher excitation amplitude F promotes frequency-locking between the forcing and self-excited oscillations, leading to stable periodic resonance and significantly enhanced EH performance.

Figure 10 presents the time histories of the vibration $x(t)$ and output power $P(t)$ picked from Fig. 9 at $F = 0.5$ obtained by using the RK4 method, confirming good agreement with the analytical findings (green lines).

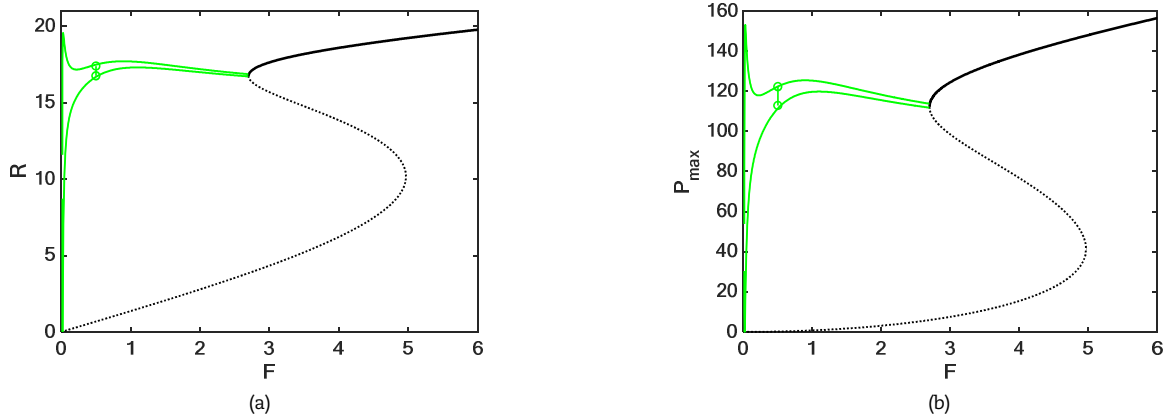


Fig. 9. Amplitude responses versus F with a vdP oscillator for $\lambda = 0.5$, $\chi = 0.2$ and $\omega = 1$, (a) Vibration, (b) Output power; Solid black (dotted) line for stable (unstable) periodic solutions; Green lines for analytical responses of QP modulation; Connected green circles for numerical simulations obtained via the RK4 method (Color online).

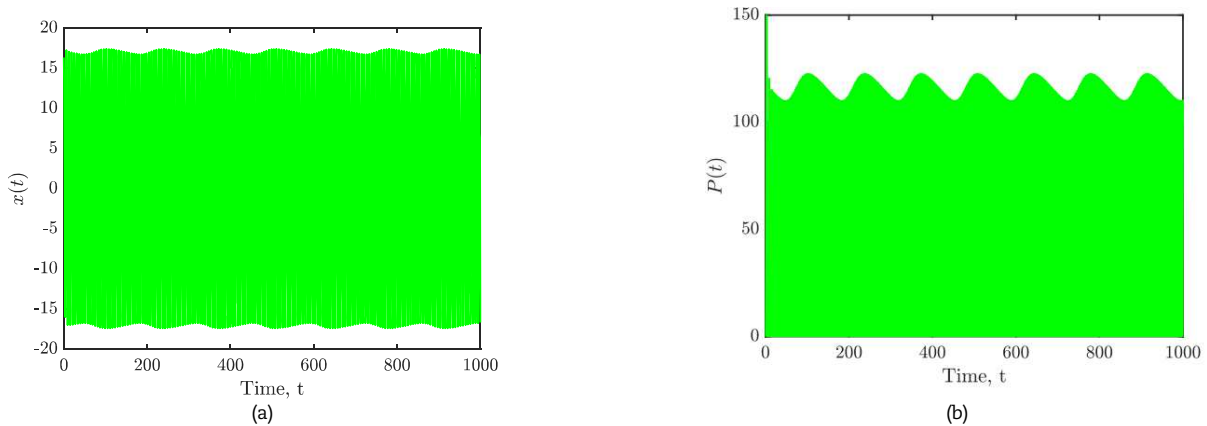


Fig. 10. Time histories of vibration $x(t)$ and (a) Output power $P(t)$, (b) Selected from Fig. 9 for $F = 0.5$ obtained by numerical integration of the governing equations with a vdP oscillator using the RK4 method (Color online).

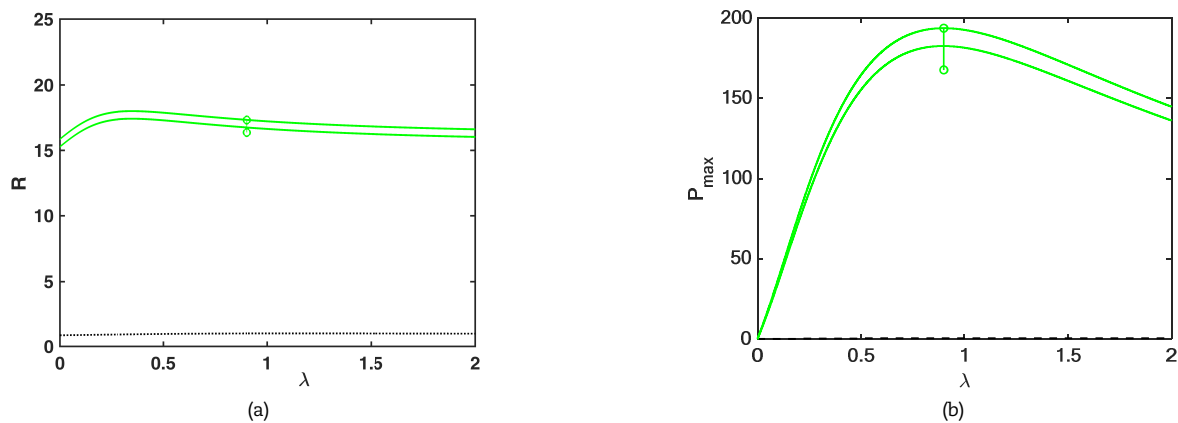


Fig. 11. Amplitude responses versus the load parameter λ with a vdP oscillator for $F = 0.7$, $\chi = 0.2$ and $\omega = 1$; (a) Vibration, (b) Output power; Dotted black line for unstable periodic solution; Green lines for analytical responses of QP modulation; Connected green circles for numerical simulations obtained via the RK4 method (Color online).



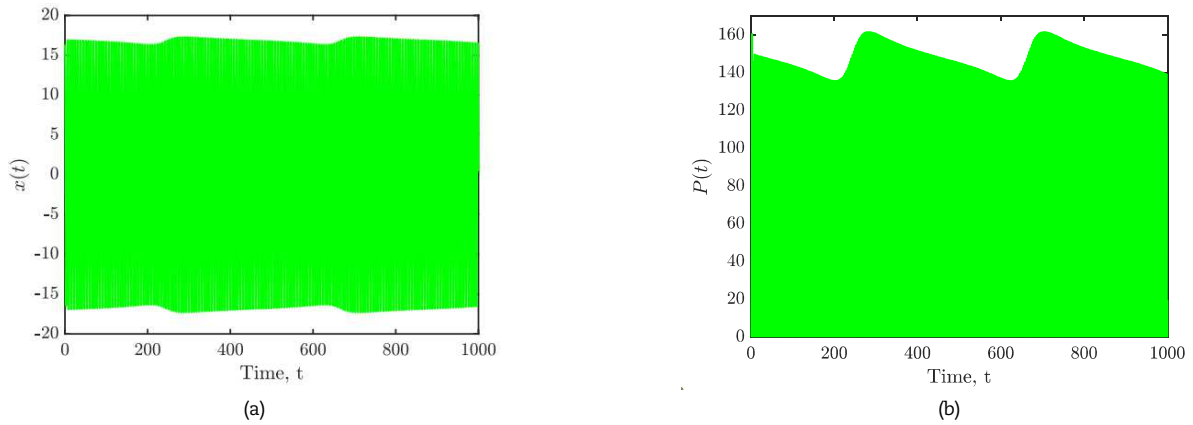


Fig. 12. Time histories of vibration $x(t)$ and (a) Output power $P(t)$, (b) Selected from Fig. 11 for $\lambda = 0.9$ obtained by numerical integration of the governing equations with a vdP oscillator using the RK4 method (Color online).

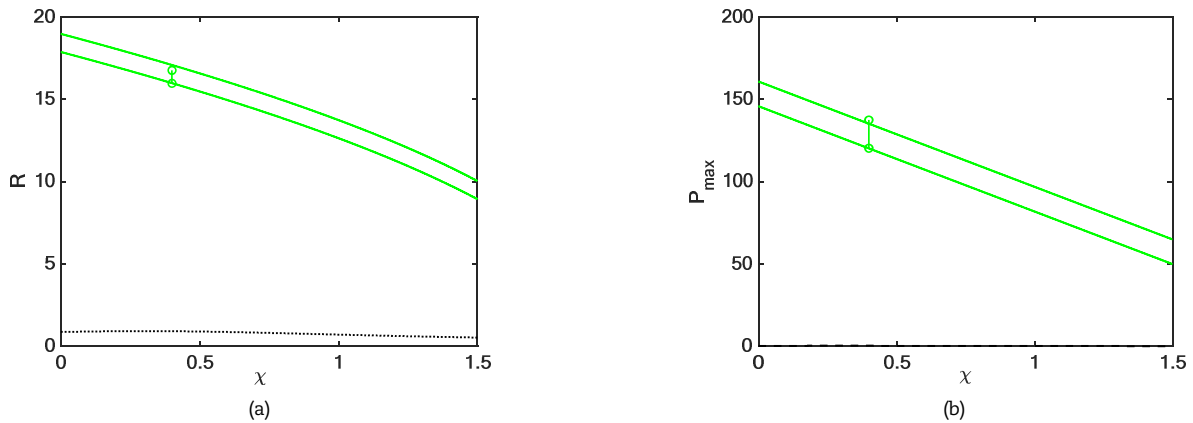


Fig. 13. Amplitude responses versus the electromechanical coupling coefficient χ with a vdP oscillator for $F = 0.7$, $\omega = 1$ and $\lambda = 0.5$; (a) Vibration, (b) Output power; Dotted blue line for unstable periodic solutions; Green lines for analytical responses of QP modulation; Connected green circles for numerical simulations obtained via the RK4 method (Color online).

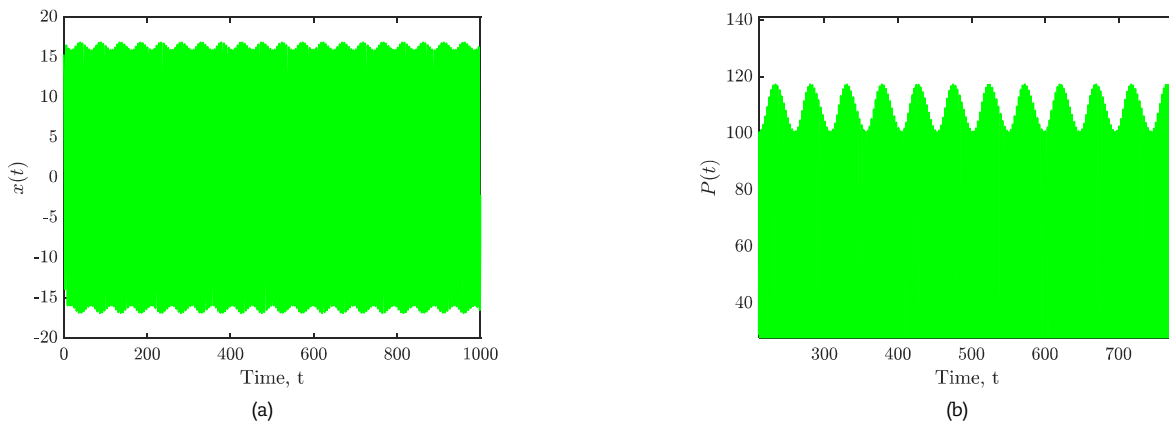


Fig. 14. Time histories of vibration $x(t)$ and (a) Output power $P(t)$, (b) Selected from Fig. 13 for $\chi = 0.4$ obtained by numerical integration of the governing equations with a vdP oscillator using the RK4 method (Color online).

The effect of the dimensionless load parameter λ is illustrated in Figs. 11(a) and 11(b), showing the amplitudes of the periodic and QP solutions of vibration and generation power, given by Eqs. (11a)–(11b), (14), and (22)–(25), with $F = 0.7$, $\chi = 0.2$, and $\omega = 1$. The connected green circles are the numerical simulation results obtained via the RK4 method. The periodic solution (dotted black line) remains unstable over the entire range of λ shown, while the QP branches (green lines) are stable and extend across a broad interval. The output power from QP vibrations attains its peak near $\lambda \approx 0.9$, reaching values of $P_{QP} \approx 193$, as shown in Fig. 11b, highlighting the effectiveness of the QP regime for EH across a wide load parameter range.

Finally, the variation of the vibration and output power amplitudes versus the electromechanical coupling coefficient χ is investigated in Figs. 13(a) and 13(b). The results are computed from Eqs. (11a)–(11b), (14), and (23)–(26) for $F = 0.7$, $\lambda = 0.5$, and $\omega = 1$. The connected green circles are the numerical simulation results obtained by the RK4 method. Over the entire range of χ considered, the periodic solution remains unstable (dotted black line), while stable QP vibrations (green lines) persist. Figure 13b shows that EH through QP dynamics is predominantly effective at small values of χ , demonstrating the ability of the system to harvest energy in the QP regime even under weak electromechanical coupling and far from the resonance region. Time histories obtained via the RK4 method confirming these observations are provided in Fig. 14 for $\chi = 0.4$.



5. Conclusion

In this paper, we investigate an electromagnetic EH system that incorporates a vdP oscillator under harmonic excitation coupled to a linear storage circuit. Analytical approximations for periodic and QP solutions of vibration and generation power are derived and analyzed using analytical methods, including the method of averaging, harmonic balance, and second-step perturbation, supplemented by numerical simulations. We examine the influence of the nonlinear damping coefficient, frequency detuning, excitation amplitude, load parameter, and electromechanical coupling coefficient on the system's response amplitude and harvested power. EH performance is evaluated across different operating conditions and compared with that of a DPLH oscillator. The study demonstrates that the self-sustained nature of the vdP oscillator can provide stable high-amplitude oscillations over a broad considered parameter range, offering advantages over conventional DPLH harvesters in terms of efficiency and performance. The findings indicate that incorporating the vdP oscillator can substantially enhance the harvested power and broaden the operating bandwidth. The system may exhibit either periodic or QP vibrations depending on the parameters. Periodic responses are favored for large excitation amplitude F and electrical load parameter λ , while QP responses dominate for small F and λ , and across a wide range of the electromechanical coupling parameter χ . The results further demonstrate that efficient EH is achievable in the QP regime over wide ranges of the considered system parameters, including regions far from resonance. These insights offer practical guidelines for designing and optimizing vibration-based electromagnetic energy harvesters utilizing self-excited oscillations to achieve broadband and efficient energy harvesting.

Author Contributions

M. Nabaoui contributed to the Data curation, Resources, Formal analysis, Investigation, Methodology, Software, Writing - Original draft; M. Hamdi contributed to the Conceptualization, Formal analysis, Investigation, Methodology, Software, Supervision, Validation, Writing - review & editing; M. Habbad contributed to the Conceptualization, Formal analysis, Investigation, Methodology, Software, Supervision, Validation, Writing - review & editing. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

Acknowledgement

Not applicable.

Conflict of Interest

The authors declare that they have no conflicts of interest related to the research, authorship, or publication of this article.

Funding

The authors received no financial support for the research, authorship and/or publication of this article.

Data Availability Statements

Data will be made available on request.

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How to cite this article: Nabaoui M., Hamdi M., Habbad M. Vibration-based energy harvesting using a van der Pol oscillator under harmonic excitation, *J. Appl. Comput. Mech.*, xx(x), 2026, 1–12. <https://doi.org/10.22055/jacm.2026.49520.6041>

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