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Using the Matrix Method to Compute the Degrees of Freedom of Mechanisms

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Abstract

In this paper, some definitions and traditional formulas for calculating the mobility of mechanisms are represented, e.g. Grubler formula, Somov - Malyshev formula, and Buchsbaum - Freudenstei. It is discussed that there are certain cases in which they are too ambiguous and incorrect to use. However, a matrix method is suggested based on the rank of the Jacobian of the mechanism and its application is investigated. It is shown that the matrix method will definitely lead to a correct answer; however, it is lengthy and consumes more computational effort. It is shown that in the cases the traditional formulas give a wrong answer and the matrix method gives the correct mobility. To compare the methods, several examples are given including the four bar planar linkage, the augmented four bar linkage, University of Maryland manipulator, Cartesian parallel manipulator (CPM), delta robot, orthoglide robot, and H4 parallel robot.

Keywords: Degrees of Freedom, Grubler's exceptions, Jacobian rank

1. Introduction

Mobility is the main parameter of a mechanism, being a fundamental issue in kinematic and dynamic model of robots and mechanisms. The number of the degrees of freedom of a mechanism is to be determined prior to any analysis or design and it is shown by the number of independent coordinates needed for precisely defining the configuration of a kinematic chain or mechanism.

Various criteria have been proposed about the analysis and calculation of degrees of the freedom of mechanisms. The mobility criterion is a relationship between the mobility, the constraint parameters of the mechanism, and the structural parameters of a mechanism such as the number of joints and links. The structural parameters are usually determined by an inspection without any need to develop the kinematic constraint equations used in mobility calculation.

Usually the following structural parameters are used in mobility criterion [1]:

1. The total number of elements including the fixed base
2. The constraint parameters
3. The number of kinematic elements
4. The number of joints
5. The degrees of constraints of the joints
6. The motion parameter
7. The number of the independent closed loops

The research on the mobility of mechanisms started in the second half of 19th century (Chebychev, 1869, Sylvester, 1874; Grübler, 1883, 1885; Somov, 1887; Hochman, 1890) and the beginning of 20th century (Koenigs, 1905;

Grübler, 1916, 1917; Malytsheff, 1923; Kutzbach, 1929). At that time the main formulas were the first Chebychev-Grubler-Kutzbach's criterion for multi loop mechanisms.

Table 1. Binary links and passive D.O.F. [2]

Type	Passive DOF
S-S	Rotation about axis through the center of sphere
S-E	Rotation about axis through the center of sphere and perpendicular to E
E-E	Sliding along axis parallel to the line of intersection of the planes of E

During the 20th century, many versions of Chebychev-Grubler-Kutzbach formula were proposed (Dobrovolski, 1949, 1951; Artobolevskii, 1953; Kolchin, 1960; Rössner, 1961; Boden, 1962; Ozol, 1963; Manolescu andManafu, 1963; Manolescu, 1968; Bagci, 1971; Hunt, 1978; Hervé, 1978; Tsai, 1999).

The mobility criterion does not fit for many mechanisms including the mechanisms proposed by De Roberval (1670), Sarrus (1853), Delassus (1900, 1902, 1922), Bennett (1903), Bricard (1927), Myard (1931), Goldberg (1943), Altman (1952), Baker (1978), Waldron (1979) et al., and many types of recently introduced parallel robots, such as Delta robot (Clavel, 1988), Star robot (Hervé and Sparacino, 1992), H4 robot (Pierrot and Company, 1999), Orthoglide (Wenger and Chablat, 2000), or CPM (Kim and Tsai, 2002).

As an example of the cases where the classic mobility formulae leads to a wrong answer, one may consider the mechanism shown in Fig. 2. The traditional method of Grubler-Kutzbach leads to a mobility of zero; where the mechanism has obviously one degree of freedom.

In this paper a new method is suggested for calculating the degrees of freedom of all types of mechanisms. This method is capable of solving the problem of complex and spatial mechanisms that may lead to a wrong answer by using a traditional method. Section 2 presents a review of frequently used formulae for computing the degrees of freedom. Section 3 proposes the new method for computing the degrees of freedom. In Section 4 some case studies are discussed.

2. Review of Traditional Formula

2.1. Grubler formula (1883)

Grubler formula is one of the oldest formulas used for computing the degrees of freedom of mechanisms:

$$F = 3(n-1) - 2j_1 - j_2 \tag{1}$$

In this formula, the number of degrees of freedom is shown by F, the number of links including the frame is shown by n, and the number of joints with *i* degree(s) of freedom is shown by j_i .

Kutzbach formula (1933);

$$F = \lambda(n - j - 1) + \sum_{i=1}^{j} f_i \tag{2}$$

In which, λ is set to 3 for planar mechanisms and 6 for spatial mechanisms, and f_i is the degrees of freedom of the i^{th} joint. There might be passive degrees of freedom in the mechanism which allow links to rotate freely about an axis defined by two joints without changing the mechanism configuration. The links would not be able to transmit forces or torques and motion about their passive axes. Table 1 shows different types of binary links and the possible passive degrees of freedom. Kutzbach formula can be modified as [3]:

$$F = \lambda(n - j - 1) - f_p + \sum_{i=1}^{j} f_i \tag{3}$$

In which f_p is the number of passive degrees of freedom of the mechanism.

2.2. Somov-Malyshev formula (1923)

According to this formula, the number of the degrees of freedom of a spatial mechanism is:

$$F = 6(l-1) - 5p_5 - 4p_4 - 3p_3 - 2p_2 - p_1 \tag{4}$$

Some interesting variations of (4) to accommodate redundant constraint mechanisms are discussed by Ruzinov [4].

2.3. Buchsbaum-Freudenstein (1970)

This formula is valid for gear driven mechanisms only [5]. Substituting in (2)

$$L_{ind} = j - l + 1 \tag{5}$$

$$j_G = L_{ind} \tag{6}$$

one obtains [6]:

$$F = j_R - j_G \tag{7}$$

Where j_R and j_G are the number of geared and revolute pairs, respectively.

3. The matrix method

Consider a finite number (n) of generalized coordinates q_k ($k = 1,..,n$), which defines the positions of all links of a mechanism. Let p be the number of independent equations that can be established between the infinitesimal variations ($\delta q_1, \delta q_2, \ldots, \delta q_n$). According to Whittaker [7], the mobility of a mechanical system is obtained as:

$$F = n - p \tag{8}$$

This expression requires the computation of p.
If the set of equations:

$$\begin{Bmatrix} \mu_1(q_1, q_2, \ldots, q_n) = 0 \\ \mu_2(q_1, q_2, \ldots, q_n) = 0 \\ . \\ . \\ . \\ \mu_m(q_1, q_2, \ldots, q_n) = 0 \end{Bmatrix} \tag{9}$$

is established between the coordinates q_k, then the following theorem holds [8]:

3.1. Theorem 1

Given m compatible functions $\mu_j (j = 1, 2, ..., m)$ in terms of n variables q_k (k=1, . . . ,n), if the rank of the Jacobian matrix is r, then there are $m - r$ relations (and not more) between the μ_j which do not involve the q_k. As a result, if the functions are independent (that is, $m = r$), there does not exist any relation between them.
The theorem just stated, supplies a criterion for testing the existence of function of the type:

$$F(\mu_1, \mu_2, ..., \mu_m) = 0 \tag{10}$$

In terms of the μ_j 's only and not the q_k 's.

Once the absence of the relations of type (10) has been ascertained, one can proceed to the computation of p. For this purpose let us partition the vector *{q}* to two parts:
Dependent coordinates:

$$\{y\} = \{y_1, y_2, \cdots, y_m\}^T$$

Independent coordinates:

$$\{x\} = \{x_1, x_2, \cdots, x_F\}^T$$

Besides, the theorem of existence of implicit functions states that [10]:

3.2. Theorem 2

Let $\mu_1, \mu_2, \cdots, \mu_m$ denote real single-valued compatible functions in terms of a finite number of n variables $(q_1, q_2, \cdots, q_n)$.

If the following conditions hold simultaneously:

$\{q^{(0)}\} = \{x_1^{(0)}, x_2^{(0)}, \cdots, x_F^{(0)}, y_1^{(0)}, y_2^{(0)}, y_m^{(0)}\}^T$ is a solution of the system of equations:

$$\begin{Bmatrix} \mu_1(q_1^{(0)}, q_2^{(0)}, \cdots, q_n^{(0)}) = 0 \\ \mu_2(q_1^{(0)}, q_2^{(0)}, \cdots, q_n^{(0)}) = 0 \\ \\ \mu_m(q_1^{(0)}, q_2^{(0)}, \cdots, q_n^{(0)}) = 0 \end{Bmatrix} \tag{11}$$

The $\mu_1, \mu_2, \cdots, \mu_m$ and all their first partial derivatives are continuous over a neighborhood $\{q^{(0)}\}$.
The determinant of the Jacobian is not zero,

$$J\left(\frac{\mu_1,\cdots,\mu_m}{y_1,\cdots,y_m}\right)=\begin{bmatrix}\frac{\partial\mu_1}{\partial y_1} & \frac{\partial\mu_1}{\partial y_2} & \cdots & \frac{\partial\mu_1}{\partial y_m}\\ \frac{\partial\mu_2}{\partial y_1} & \frac{\partial\mu_2}{\partial y_2} & \cdots & \frac{\partial\mu_2}{\partial y_m}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial\mu_m}{\partial y_1} & \frac{\partial\mu_m}{\partial y_2} & \cdots & \frac{\partial\mu_m}{\partial y_m}\end{bmatrix} \tag{12}$$

Then within a neighborhood of $\left\{q^{(0)}\right\}$, the equations (9), would define $(y_1,y_2,\cdots,y_m)$ as single-valued functions of $(x_1,x_2,\cdots,x_F)$.

If the conditions mentioned by theorem 2 are all satisfied, then

$$p=m$$

And otherwise

$$p=r$$

Where r is the rank of Jacobian (12).

The matrix approach can prove how geometry, link positions, and input link, affect the number of the degrees of freedom of a mechanism. Since the criterion discussed in this section requires the evaluation of the derivatives of the constraint equations, the conclusions on the number of the degrees of freedom are limited to a given configuration and limited to infinitesimal displacements. To use it globally, one should evaluate the formulas symbolically or point-wise in the whole workspace wherever needed. The configurations of a mechanism without a full rank of the Jacobian, are named singular. If rank deficiency is maintained for a finite range of movements, then the singular form is said to be permanent, and otherwise instantaneous.

4. Case Study

Here in, a number of mechanisms are considered and the applications of the methods are investigated and compared.

4.1. The four bar planar linkage

As a simple example, a four bar planar mechanism is considered in Fig. 1. The independent loop equations are:

$$\{\mu\}=\begin{Bmatrix}AB\cos\alpha+BC\cos\beta+CD\cos\gamma-L\\ AB\sin\alpha+BC\sin\beta+CD\sin\gamma-H\end{Bmatrix} \tag{13}$$

The difference between the number of joints and the number of links is $L-1$. Hence, the number of independent loops in the mechanism is [2]:

$$L=j-n+1 \tag{14}$$

To implement theorem 2, one should note that α, β, and γ are the parameters of the mechanism which are employed in (13). It is observed that μ and its first partial derivatives are continuous. The Jacobian of μ is:

$$J\left(\frac{\mu_1,\mu_2}{\beta,\gamma}\right)=\begin{bmatrix}-BC\sin\beta & -CD\sin\gamma\\ BC\cos\beta & CD\cos\gamma\end{bmatrix} \tag{15}$$

Note that the determinant of Jacobian (15) is not zero in the general case (except for singularities) and according to the theorem and the aforementioned conditions:

$$p=m=2 \tag{16}$$

As there were three coordinates considered for this mechanisms, applying equation (8) gives $F=1$, i.e. a one degree of freedom mechanism.

4.2. Augmented 4-Bar Linkage

In this case, a counter example of Grubler's law is considered. In Fig. 2, links AB, CD, and EF are parallel and equal in length. Both Grubler and Kuztbach formulas lead to wrong answers for the degrees of freedom of this mechanism. To apply Grubler formula the number of the links including the fixed base, n , and the number of the joints with i degrees of freedom, j_i , are:

$$\begin{aligned} j_1&=6\\ n&=5\end{aligned} \tag{17}$$

So one may obtain using Grubler formula:

$$F = 3(5-1) - 2(6) = 0 \tag{18}$$

Which is a wrong answer for the degrees of freedom of this mechanism [2].
To apply Kutzabach formula (3), the motion number is $\lambda = 3$ for the planar mechanism, the number of the passive joints is $f_p = 0$, the sum of the degree of freedom of all joints is $\sum_{i=1}^{6} f_i = 6$, and the number of the degrees of freedom of the mechanism is thus:

$$F = 3(5-6-1) - 0 + 6 = 0 \tag{19}$$

This also leads to an incorrect answer.
To compute the correct number of the degrees of freedom, one should use (13) considering two independent loop equations:

$$\vec{\mu} = \begin{Bmatrix} \overrightarrow{AB} + \overrightarrow{BD} - \overrightarrow{CD} - \overrightarrow{AC} \\ \overrightarrow{CD} + \overrightarrow{DF} - \overrightarrow{FE} - \overrightarrow{EC} \end{Bmatrix} \tag{20}$$

The dependent and independent variables are:

$$\begin{aligned} \{x\} &= \{x_1 = \theta\} \\ \{y\} &= \{0\} \end{aligned} \tag{21}$$

Now by considering the independent and dependent variables, rewriting (20) yields:

$$\{\mu\} = \begin{Bmatrix} AB\cos\theta + BD - DC\cos\theta - E1 \\ AB\sin\theta - DC\sin\theta \\ CD\cos\theta + DF - FE\cos\theta - E2 \\ CD\sin\theta - FE\sin\theta \end{Bmatrix} \tag{22}$$

In which *E1* and *E2* are constants. As the lengths of *AB*, *DC* and *FE* are equal, μ becomes:

$$\{\mu\} = \begin{Bmatrix} BD - E1 \\ 0 \\ DF - E2 \\ 0 \end{Bmatrix} \tag{23}$$

And the Jacobian of μ:

$$J\left(\frac{\partial \mu}{\partial y}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

Here, the Jacobian of μ is zero making the rank of Jacobian matrix zero. So, applying theorem 2, the number of the degrees of freedom of this mechanism is computed:

$$\begin{aligned} n &= 1 \\ p &= r = 0 \\ F &= 1 - 0 = 1 \end{aligned} \tag{25}$$

which is the correct answer.

4.3. University of Maryland Manipulator

Another counter example of Grubler's law is University of Maryland manipulator represented in Fig. 3. This mechanism employs only revolute joints. To apply Kutzbach formula (3):

$$\begin{aligned} &\lambda = 6 \\ &n = 17 \\ &j = 21 \\ &f_p = 0 \\ &\sum_{i=1}^{21} f = 21 \end{aligned} \tag{26}$$

$$F = 6(17 - 21 - 1) + 21 - 0 = -9 \tag{27}$$

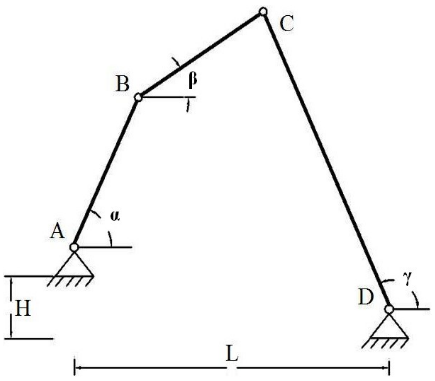

Fig. 1. Parameters of a four bar planar linkage (case study 4.1.)

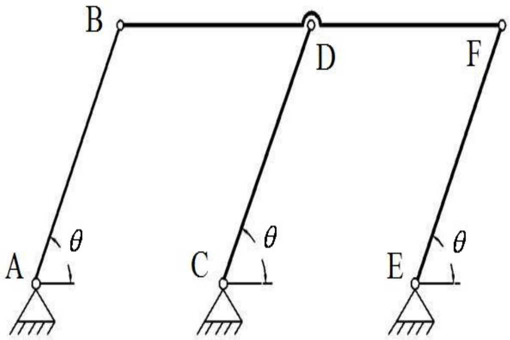

Fig. 2. Parameters of an augmented four bar planar linkage(case study 4.2.)

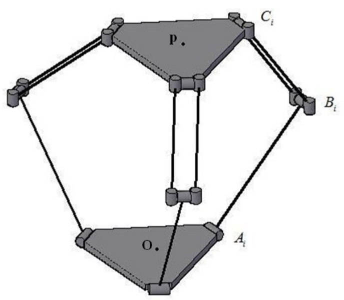

Fig. 3. University of Maryland manipulator scheme

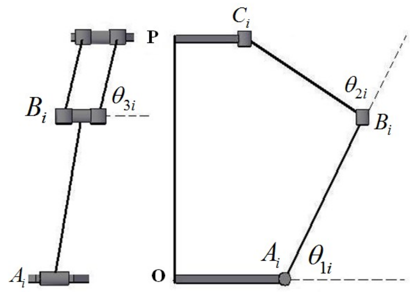

Fig. 4. The definitions of the angles of University of Maryland manipulator

Using the proposed matrix method, one may compute the correct number of the degrees of freedom. As the first step, the independent loop equations are:

$$\mu = \begin{Bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{Bmatrix} = \begin{Bmatrix} \overrightarrow{A_1B_1} + \overrightarrow{B_1C_1} - \overrightarrow{C_1P} - \overrightarrow{PO} + \overrightarrow{OA_1} \\ \overrightarrow{A_2B_2} + \overrightarrow{B_2C_2} - \overrightarrow{C_2P} - \overrightarrow{PO} + \overrightarrow{OA_2} \\ \overrightarrow{A_3B_3} + \overrightarrow{B_3C_3} - \overrightarrow{C_3P} - \overrightarrow{PO} + \overrightarrow{OA_3} \end{Bmatrix} \tag{28}$$

The dependent and independent variables are:

$$\begin{aligned} \{x\} &= \{\theta_{21}, \theta_{31}, \theta_{22}, \theta_{32}, \theta_{23}, \theta_{33}\} \\ \{y\} &= \{\theta_{11}, \theta_{12}, \theta_{13}\} \end{aligned} \tag{29}$$

Considering Fig. 4, the geometric relations inferred are:

$$\begin{aligned} A_1B_1 &= A_2B_2 = A_3B_3 = AB \\ B_1C_1 &= B_2C_2 = B_3C_3 = BC \end{aligned} \tag{30}$$

Eventually, considering the dependent and independent variables (29), and the geometric relations (30), one may rewrite (28):

$$\mu_1 = \begin{Bmatrix} A_1B_1\cos\theta_{11} + B_1C_1\sin\theta_{31}\cos(\theta_{11}+\theta_{21}) - \overrightarrow{C_1P} + \overrightarrow{OA_1} + p_x \\ B_1C_1\cos\theta_{31} - p_y \\ A_1B_1\sin\theta_{11} + B_1C_1\sin\theta_{31}\sin(\theta_{11}+\theta_{21}) - p_z \end{Bmatrix} \tag{31}$$

Note that μ and the first derivatives are continuous. The Jacobian matrix of μ is:

$$J\left(\frac{\overrightarrow{\mu_1},\overrightarrow{\mu_2},\overrightarrow{\mu_3}}{\theta_{21},\theta_{31},\theta_{22},\theta_{32},\theta_{23},\theta_{33}}\right) \tag{32}$$

Using theorem 2 and the conditions mentioned before, the rank of the Jacobian is computed to be:

$$p=r=6 \tag{33}$$

And using (8), the number of the degrees of freedom of this mechanism is calculated to be:

$$F=9-6=3 \tag{34}$$

4.4. Cartesian Parallel Manipulator (CPM)

The CPM is one of the famous counter examples of Grubler's law. The CPM under study is shown in Fig.5. In this manipulator three PRRR mechanisms are used in parallel. To apply Kutzbach formula (3):

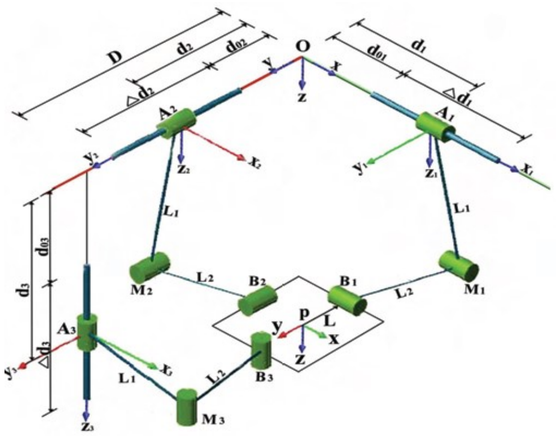

Fig. 5. Cartesian Parallel manipulator Scheme

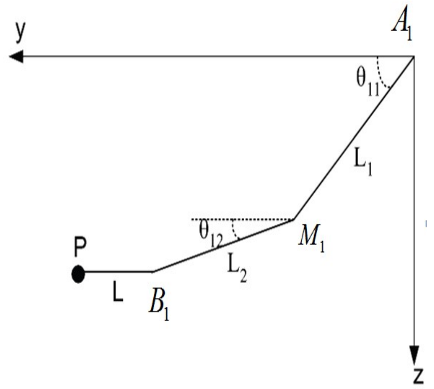

Fig. 6. Definition of angles of the robot shown in Fig. 5

$$\begin{aligned} &\lambda=6\\ &n=11\\ &j=12\\ &f_p=0\\ &\sum_{i=1}^{12} f=12 \end{aligned} \tag{35}$$

Hence using this formula, the number of the degrees of freedom is computed as:

$$F=6(11-12-1)+12-0=0 \tag{36}$$

Now, to use the matrix method to calculate the number of the degrees of freedom of the CPM, one may first write the independent loop equations:

$$\mu=\begin{Bmatrix}\overrightarrow{\mu_1}\\ \overrightarrow{\mu_2}\\ \overrightarrow{\mu_3}\end{Bmatrix}=\begin{Bmatrix}\overline{OA_1}+\overline{A_1M_1}+\overline{M_1B_1}+\overline{B_1P}-\overline{PO}\\ \overline{OA_2}+\overline{A_2M_2}+\overline{M_2B_2}+\overline{B_2P}-\overline{PO}\\ \overline{OA_3}+\overline{A_3M_3}+\overline{M_3B_3}+\overline{B_3P}-\overline{PO}\end{Bmatrix} \tag{37}$$

In this mechanism, the dependent and independent variables are, respectively:

$$\begin{aligned} &\{y\}=\{\theta_{11},\theta_{12},\theta_{21},\theta_{22},\theta_{31},\theta_{32}\}\\ &\{x\}=\{d_1,d_2,d_3\} \end{aligned} \tag{38}$$

Considering Fig. 6, the geometric relations are defined:

$$\begin{aligned} &A_1M_1=A_2M_2=A_3M_3=l_1\\ &M_1B_1=M_2B_2=M_3B_3=l_2\\ &OA_1=d_1, OA_2=d_2, OA_3=d_3\\ &B_1P=B_2P=B_3P=l \end{aligned} \tag{39}$$

Now by considering the dependent and independent variables (38) and the geometric relations (39), one may rewrite (37) as:

$$\mu_1 = \begin{Bmatrix} d_1 - P_x \\ l_1 COS\,\theta_{11} + l_2 COS\,(\theta_{11} + \theta_{12}) + l - P_y \\ l_1 SIN\,\theta_{11} + l_2 SIN\,(\theta_{11} + \theta_{12}) - P_z \end{Bmatrix} \tag{40}$$

$$\mu_2 = \begin{Bmatrix} l_1 COS\,\theta_{21} + l_2 COS\,(\theta_{21} + \theta_{22}) + l - P_x \\ d_2 - P_y \\ l_1 SIN\,\theta_{21} + l_2 SIN\,(\theta_{21} + \theta_{22}) - P_z \end{Bmatrix} \tag{41}$$

$$\mu_3 = \begin{Bmatrix} l_1 COS\,\theta_{31} + l_2 COS\,(\theta_{31} + \theta_{32}) - P_x \\ d - l_1 SIN\,\theta_{31} - l_2 SIN\,(\theta_{31} + \theta_{32}) - l - P_y \\ d_3 - P_z \end{Bmatrix} \tag{42}$$

Note that, μ and its first derivatives are continuous. Thus the Jacobian matrix of μ is:

$$J\left(\frac{\overrightarrow{\mu_1}, \overrightarrow{\mu_2}, \overrightarrow{\mu_3}}{\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}, \theta_{31}, \theta_{32}}\right) \tag{43}$$

Using theorem 2 and the conditions that mentioned before, the rank of the Jacobian is computed to be:

$$p=r=6 \tag{44}$$

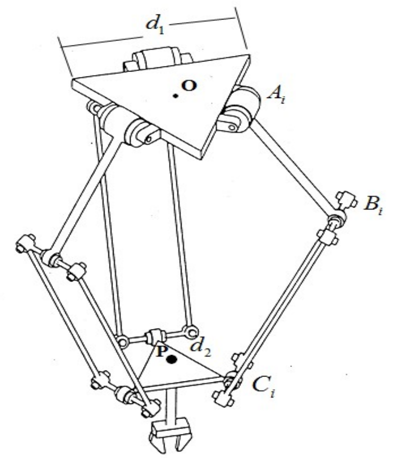

Fig. 7. Delta Robot Scheme

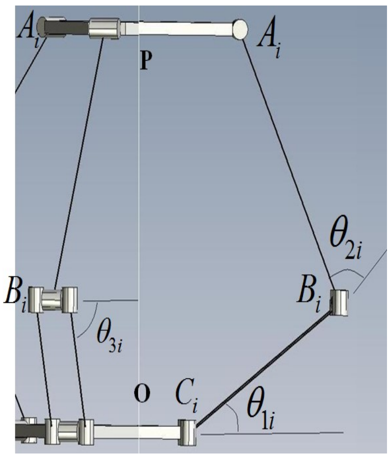

Fig. 8. Definition of angles of the robot shown in Fig.7

Thus using equation (8), the number of the degrees of freedom of Cartesian Parallel Manipulator (CPM) is calculated to be: F=9-6=3.

4.5. Delta Robot

Another famous counter example of Grubler's law is Delta Robot [2] as depicted in Fig. 7. This mechanism employs revolute joints only. To apply Kutzabch formula (3) using parameter set:

$$\begin{aligned} &\lambda = 6 \\ &n = 17 \\ &j = 21 \\ &f_p = 0 \\ &\sum_{i=1}^{21} f = 21 \end{aligned} \tag{45}$$

one will be led to an incorrect answer:

$$F = 6(17-21-1)+21-0 = -9 \tag{46}$$

To use the proposed method, the first step to calculate the number of the degrees of the freedom is to write independent loop equations:

$$\mu = \begin{Bmatrix} \overrightarrow{\mu_1} \\ \overrightarrow{\mu_2} \\ \overrightarrow{\mu_3} \end{Bmatrix} = \begin{Bmatrix} \overrightarrow{A_1B_1} + \overrightarrow{B_1C_1} - \overrightarrow{C_1P} - \overrightarrow{PO} + \overrightarrow{OA_1} \\ \overrightarrow{A_2B_2} + \overrightarrow{B_2C_2} - \overrightarrow{C_2P} - \overrightarrow{PO} + \overrightarrow{OA_2} \\ \overrightarrow{A_3B_3} + \overrightarrow{B_3C_3} - \overrightarrow{C_3P} - \overrightarrow{PO} + \overrightarrow{OA_3} \end{Bmatrix} \quad (47)$$

In this mechanism the dependent and independent variables are:

$$\begin{aligned} \{y\} &= \{\theta_{12}, \theta_{13}, \theta_{22}, \theta_{23}, \theta_{32}, \theta_{33}\} \\ \{x\} &= \{\theta_{11}, \theta_{21}, \theta_{31}\} \end{aligned} \quad (48)$$

Considering Fig. 8, the geometric relations may be defined as:

$$\begin{aligned} A_1B_1 &= A_2B_2 = A_3B_3 = a \\ B_1C_1 &= B_2C_2 = B_3C_3 = b \\ OA_1 &= OA_2 = OA_3 = d_1 \\ B_1P &= B_2P = B_3P = d_2 \end{aligned} \quad (49)$$

By considering the dependent and independent variables (48) and the geometric relations (49), equations (47) can be simplified to:

$$\mu_1 = \begin{Bmatrix} -aCOS(\theta_{11}) - bCOS(\theta_{11} + \theta_{12})SIN(\theta_{13}) - P_x \\ -\frac{\sqrt{3}}{6}d_1 + \frac{\sqrt{3}}{3}d_2 - bCOS(\theta_{13}) - P_y \\ -aSIN(\theta_{11}) - bSIN(\theta_{11} + \theta_{12})SIN(\theta_{13}) - P_z \end{Bmatrix} \quad (50)$$

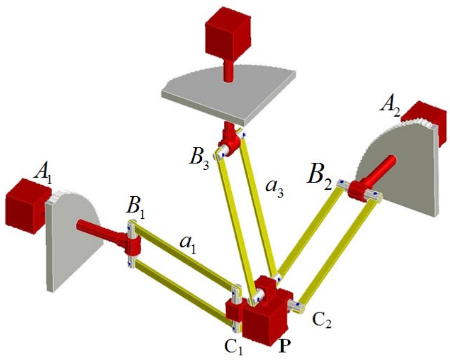

Fig. 9. Orthoglide Robot Scheme

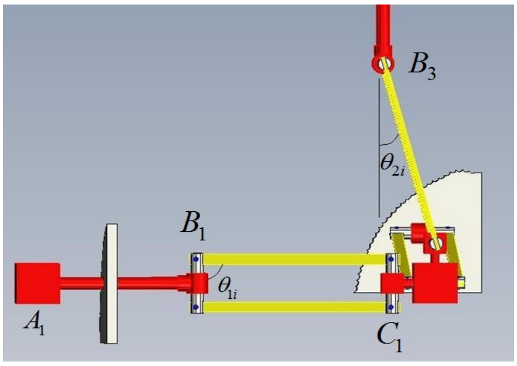

Fig. 10. Definition of angles of the robot shown in Fig.9

$$\mu_2 = \begin{Bmatrix} -aCOS(\theta_{21}) - bCOS(\theta_{21} + \theta_{22})SIN(\theta_{23}) - P_x + \frac{d_1}{4} - \frac{d_2}{2} \\ -\frac{\sqrt{3}}{6}d_1 - \frac{\sqrt{3}}{3}d_2 - bCOS(\theta_{23}) - P_y \\ -aSIN(\theta_{21}) - bSIN(\theta_{21} + \theta_{22})SIN(\theta_{23}) - P_z \end{Bmatrix} \quad (51)$$

$$\mu_3 = \begin{Bmatrix} -aCOS(\theta_{31}) - bCOS(\theta_{31} + \theta_{32})SIN(\theta_{33}) - P_x - \frac{d_1}{4} + \frac{d_2}{2} \\ \frac{\sqrt{3}}{12}d_1 - \frac{\sqrt{3}}{6}d_2 - bCOS(\theta_{33}) - P_y \\ -aSIN(\theta_{31}) - bSIN(\theta_{31} + \theta_{32})SIN(\theta_{33}) - P_z \end{Bmatrix} \quad (52)$$

Note that, μ and its first derivatives are continuous. Thus the Jacobian matrix of μ is:

$$J\left(\frac{\overrightarrow{\mu_1}, \overrightarrow{\mu_2}, \overrightarrow{\mu_3}}{\theta_{12}, \theta_{13}, \theta_{22}, \theta_{23}, \theta_{32}, \theta_{33}}\right) \quad (53)$$

Using theorem 2, the rank of the Jacobian is computed to be:

$$p=r=6 \quad (54)$$

Hence the number of the degrees of freedom of this mechanism is calculated to be: F=9-6=3.

4.6. Orthoglide Robot

The Orthoglide Robot is another famous counter example of Grubler's law. As depicted in Fig. 9, this mechanism employs revolute joints only. Employing Kutzabch formula (3) with the parameter set:

$$\begin{aligned} &\lambda = 6 \\ &n = 14 \\ &j = 18 \\ &f_p = 0 \\ &\sum_{i=1}^{21} f = 18 \end{aligned} \tag{55}$$

will lead to a wrong number of the degrees of freedom:

$$F = 6(14 - 18 - 1) + 18 - 0 = -12 \tag{56}$$

Using the proposed matrix method, the first step is to write the independent loop equations:

$$\mu = \begin{Bmatrix} \overrightarrow{\mu_1} \\ \overrightarrow{\mu_2} \\ \overrightarrow{\mu_3} \end{Bmatrix} = \begin{Bmatrix} \overrightarrow{A_1B_1} + \overrightarrow{B_1C_1} + \overrightarrow{C_1P} - \overrightarrow{PO} + \overrightarrow{OA_1} \\ \overrightarrow{A_2B_2} + \overrightarrow{B_2C_2} + \overrightarrow{C_2P} - \overrightarrow{PO} + \overrightarrow{OA_2} \\ \overrightarrow{A_3B_3} + \overrightarrow{B_3C_3} + \overrightarrow{C_3P} - \overrightarrow{PO} + \overrightarrow{OA_3} \end{Bmatrix} \tag{57}$$

In this mechanism the dependent and independent variables are:

$$\begin{aligned} &\{y\} = \{\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22}, \theta_{13}, \theta_{23}\} \\ &\{x\} = \{d_x, d_y, d_z\} \end{aligned} \tag{58}$$

Considering Fig. 10, the geometric relations may be inferred:

$$\begin{aligned} &A_1B_1 = d_1, A_2B_2 = d_2, A_3B_3 = d_3 \\ &B_1C_1 = a_1, B_2C_2 = a_2, B_3C_3 = a_3 \\ &C_1P = C_2P = C_3P = p \end{aligned} \tag{59}$$

Now by considering the dependent and independent variables (58) and the geometric relations (59), one my rewrite (57) as:

$$\mu_1 = \begin{Bmatrix} -a_1COS(\theta_{11})SIN(\theta_{11}) - P_1COS(\theta_{11} + \theta_{21}) - P_x \\ a_1COS^2(\theta_{11}) + P_1COS(\theta_{11} + \theta_{21}) + d_1 - P_y \\ -a_1SIN(\theta_{11}) - P_z \end{Bmatrix} \tag{60}$$

$$\mu_2 = \begin{Bmatrix} a_2COS^2(\theta_{12}) - P_2COS(\theta_{12} + \theta_{22}) + d_2 \\ \quad - P_x \\ -a_2SIN(\theta_{12}) - P_y \\ -a_2COS(\theta_{12})SIN(\theta_{12}) - P_2COS(\theta_{12} + \theta_{22}) \\ \quad - P_z \end{Bmatrix} \tag{61}$$

$$\mu_3 = \begin{Bmatrix} -a_3SIN(\theta_{12}) - P_x \\ a_3COS^2(\theta_{13}) - P_3COS(\theta_{13} + \theta_{23}) - P_y \\ -a_3COS(\theta_{13})SIN(\theta_{13}) - P_3SIN(\theta_{13} + \theta_{23}) \\ \quad + d_3 - P_z \end{Bmatrix} \tag{62}$$

Note that, μ and its first derivatives are continuous. Thus, using theorem 2, the rank of the Jacobian matrix of μ:

$$J\left(\frac{\overrightarrow{\mu_1}, \overrightarrow{\mu_2}, \overrightarrow{\mu_3}}{\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22}, \theta_{13}, \theta_{23}}\right) \tag{63}$$

is computed to be:

$$p = r = 6 \tag{64}$$

So, the number of the degrees of freedom of this mechanism is calculated from (8) to be: F=9-6=3.

4.7. H4 Parallel Robot

Another famous counter example of Grubler's law is H4 Parallel Manipulator as shown in Fig. 11. To apply Kutzabch formula (3), using the parameter set:

$$\begin{aligned} &\lambda = 6 \\ &n = 24 \\ &j = 30 \\ &f_p = 0 \\ &\sum_{i=1}^{21} f = 30 \end{aligned} \tag{65}$$

will lead to an incorrect number of the degrees of freedom:

$$F = 6(24-30-1)+30-0 = -12 \tag{66}$$

To calculate the correct number of the degrees of freedom using the matrix method, one may write the independent loop equations:

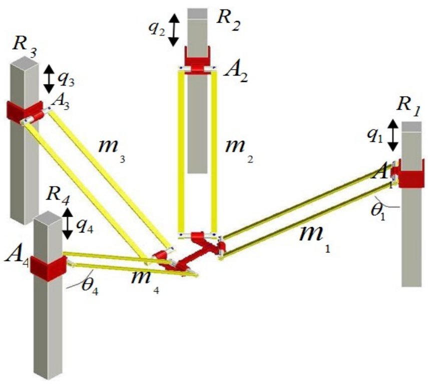

Fig. 11. H4 Parallel Manipulator Scheme

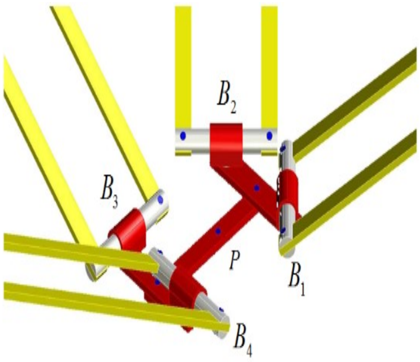

Fig. 12. Detailed definition of the end-effector of the robot shown in Fig. 11

$$\mu = \begin{Bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{Bmatrix} = \begin{Bmatrix} \overrightarrow{A_1B_1} + \overrightarrow{B_1C_1} + \overrightarrow{C_1P} - \overrightarrow{PO} + \overrightarrow{OR_1} - \overrightarrow{R_1A_1} \\ \overrightarrow{A_2B_2} + \overrightarrow{B_2C_1} + \overrightarrow{C_1P} - \overrightarrow{PO} + \overrightarrow{OR_2} - \overrightarrow{R_2A_2} \\ \overrightarrow{A_3B_3} + \overrightarrow{B_3C_2} + \overrightarrow{C_2P} - \overrightarrow{PO} + \overrightarrow{OR_3} - \overrightarrow{R_3A_3} \\ \overrightarrow{A_4B_4} + \overrightarrow{B_4C_2} + \overrightarrow{C_2P} - \overrightarrow{PO} + \overrightarrow{OR_4} - \overrightarrow{R_4A_4} \end{Bmatrix} \tag{67}$$

In this mechanism the dependent and independent variables are:

$$\begin{aligned} &\{y\} = \{\theta_1, \theta_2, \theta_3, \theta_4, \alpha_1, \alpha_2, \beta, \gamma, \varphi\} \\ &\{x\} = \{d_1, d_2, d_3, d_4\} \end{aligned} \tag{68}$$

α_1 and α_2 are the angles between limb "d" and XY plane. φ is the angle between limb "H" and XY plane. β and γ are the angles the limbs "d" and "H" make with X direction, respectively.

Considering Fig. 12, the geometric parameters are defined as:

$$
\begin{aligned}
&R_1A_1 = q_1, R_2A_2 = q_2 \\
&R_3A_3 = q_3, R_3A_3 = q_3 \\
&A_1B_1 = m_1, A_2B_2 = m_2 \\
&A_3B_3 = m_3, A_4B_4 = m_4 \\
&B_1C_1 = B_2C_2 = B_3C_3 = B_4C_4 = d \\
&C_1P = C_2P = h
\end{aligned} \tag{69}
$$

Now by considering the variables (68), the geometric relations (69), one my rewrite (67) as:

$$
\mu_1 = \begin{Bmatrix} m_1SIN(\theta_1) + dCOS(\alpha_1)COS(\beta) + hCOS(\varphi)COS(\gamma) - P_x \\ hCOS(\varphi)COS(\gamma) + dCOS(\alpha_1)SIN(\beta) - P_y \\ q_1 + m_1COS(\theta_1) + dSIN(\alpha_1) + hSIN(\varphi) - P_z \end{Bmatrix} \tag{70}
$$

$$
\mu_2 = \begin{Bmatrix} m_2SIN(\theta_2) + dCOS(\alpha_1)COS(\beta) + hCOS(\varphi)SIN(\gamma) - P_x \\ hCOS(\varphi)COS(\gamma) - dCOS(\alpha_1)SIN(\beta) - P_y \\ q_2 + m_2COS(\theta_2) - dSIN(\alpha_1) + hSIN(\varphi) - P_z \end{Bmatrix} \tag{71}
$$

$$
\mu_3 = \begin{Bmatrix} m_3SIN(\theta_3) + dCOS(\alpha_2) - hCOS(\varphi)SIN(\gamma) - P_x \\ -hCOS(\varphi)COS(\gamma) - P_y \\ q_3 + m_3COS(\theta_3) + dSIN(\alpha_2) - hSIN(\varphi) - P_z \end{Bmatrix} \tag{72}
$$

$$
\mu_4 = \begin{Bmatrix} m_4SIN(\theta_4) - hCOS(\varphi)SIN(\gamma) - P_x \\ -dCOS(\alpha_2) - hCOS(\varphi)COS(\gamma) - P_y \\ q_4 + m_4COS(\theta_4) - dSIN(\alpha_2) - hSIN(\varphi) - P_z \end{Bmatrix} \tag{73}
$$

Note that, μ and its first derivatives are continuous. Thus the Jacobian matrix of μ is:

$$
J\left(\frac{\mu_1, \mu_2, \mu_3, \mu_4}{\theta_1, \theta_2, \theta_3, \theta_4, \alpha_1, \alpha_2, \beta, \gamma, \varphi} \right) \tag{74}
$$

Using theorem 2 and the conditions mentioned earlier, the rank of the Jacobian is computed to be:

$$
p = r = 9 \tag{2}
$$

Thus, the number of the degrees of freedom of this mechanism is calculated from (8) to be: F=13-9=4.

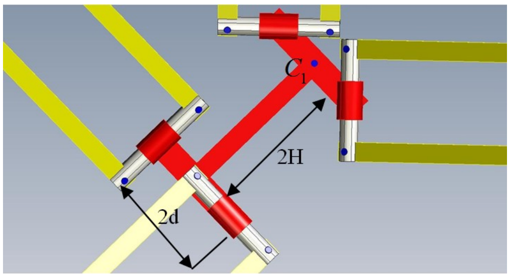

Fig. 13. Definition of angles of the robot shown in Fig. 11

5. Conclusion

Different traditional approaches to obtain the number of the degrees of freedom of mechanisms were discussed and a new method using matrix approach was explained. Using several case studies, it was shown that this general method can be used to obtain the correct number of degrees of freedom of any mechanism employing the governing loop equations. This is especially useful as the common methods like Grubler and Kutzbach may lead to incorrect answers in some special types of mechanisms without knowing that the computed number of the degrees of freedom is not correct. The methods were compared upon multiple examples of different types of mechanisms. The matrix method always leads to the correct answer, however, it is more complex and takes more computational efforts compared to other methods.

References

[1] Gogu, G. "Chebychev–Grübler–Kutzbach's criterion for mobility calculation of multi-loop mechanisms revisited via theory of linear transformations." *European Journal of Mechanics-A/Solid*s 24, 3, pp. 427-441, 2005.

[2] Tsai, L.W. "The mechanics of Serial and Parallel manipulators". New York, NY: John Wiley and Sons, ISBN 0-471-32593-7, 1999.

[3] Tsai, L.W. "Mechanism design: enumeration of kinematic structures according to function". CRC press, 2000.

[4] Ruzinov, L.D. "Design of Mechanisms by Geometric Transformations. Iliffe, 1968.

[5] Buchsbaum, F., Freudenstein, F. "Synthesis of kinematic structure of geared kinematic chains and other mechanisms." *Journal of Mechanisms* 5, 3, pp. 357-392, 1970.

[6] Paul, B. "A unified criterion for the degree of constraint of plane kinematic chains" *Journal of Applied Mechanics* 27, 1, pp. 196-200, 1960.

[7] Whittaker, E.T. "A treatise on the analytical dynamics of particles and rigid bodie"s. Cambridge University Press, 1988.

[8] Levi-Civita, T. "The absolute differential calculus (calculus of tensors)." Courier Corporation, 1926. Reprint by Dover Publications, 1977.

[9] Freudenstein, F. "On the variety of motions generated by mechanisms." *Journal of Engineering for Industry* 84(1), pp.156-159, 1962.

[10] Litvin, F.L. "Application of theorem of implicit function system existence for analysis and synthesis of linkages" *Mechanism and Machine Theory* 15(2), pp. 115-125, 1980.

[11] Burton, P., Huston, R.L. "Kinematics and Dynamics of Planar Machinery." *Journal of Applied Mechanics* 47, 459, 1980.

[12] Di Benedetto, A., Pennestrı, E. "Introduction to Mechanism Kinematic"s. Casa Editrice Ambrosiana in Italian, 1993.