A Simple Approach to Volterra-Fredholm Integral Equations

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Abstract. This paper suggests a simple analytical method for Volterra-Fredholm integral equations, the solution process is similar to that by variational-based analytical method, e.g., Ritz method, however, the method requires no establishment of the variational principle for the discussed problem, making the method much attractive for practical applications. The examples show the method is straightforward and effective, and the method can also be extended to other nonlinear problems.

Keywords: Integral equation, Series solution, Variational principle.

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1. Introduction

As integral equations arise everywhere from the architectural engineering to nanotechnology [1-5], analytical methods for such problems have been caught much attention. Ghorbani and Saberi-Nadjafi suggested a modification of the homotopy perturbation method [6], Novin & Araghi also suggested a modified homotopy perturbation method for hypersingular integral equations [7], Deniz gave an optimal perturbation iteration technique [8], which was a development of the iteration perturbation method [9]. Tian found that the Monte Carlo method is an effective tool to integral equations [10, 11]. A review on various analytical methods for integral equations is available in Ref. [12]. This paper focuses on Volterra-Fredholm integral equations, and a simple analytical method is suggested.

2. Volterra-Fredholm Integral Equations

This paper adopts two examples in Ref. [8] to show the solution process.

\[ y(x) = g(x) + \int_0^x f(x,s)ds + \int_0^1 h(x,s)ds \quad 0 \leq x \leq 1 \]  

(1)

Similar to various analytical methods in the variational theory [13, 14], e.g., the Ritz method, we can choose to a suitable trial solution for the problem. The most used trial solution is the series form

\[ y(x) = \sum_{n=0}^N a_n x^n \]  

(2)

where \( a_n \) (n=0–N) are unknown constants.

Submitting Eq. (2) and expanding \( g(x) \) into a series of \( x \), combining the like terms, and setting the coefficients of \( x^n \) (n=0–N) to zero, we obtain algebraic equations for \( a_n \) (n=0–N).

Alternatively, we can assume the solution has the form

\[ y(x) = \sum_{n=0}^N A_n(x) \]  

(3)
where \( A_n(x) \) \((n=0\sim N)\) are unknown functions. The choice of \( A_n(x) \) depends upon the \( g(x) \). For example \( g(x) = e^x \), we can assume the solution has the form \( y(x) = a_0 + a_1 e^x + a_2 e^{2x} + \cdots \), where \( a_n \) are unknown constants.

**Example 1** [8]. Consider the following Volterra-Fredholm integral equation

\[
y(x) = -\frac{1}{30}x^5 + \frac{1}{2}x^4 - x^2 + \frac{5}{3}x - \frac{5}{4} + \int_0^x (x - s)y(s)ds + \int_0^x (x + s)y(s)ds
\]

To elucidate the solution process, we assume the approximate solution can be expressed as

\[
y(x) = a_0 + a_1 x + a_2 x^2
\]

Putting Eq. (5) into Eq. (4) results in

\[
a_0 + a_1 x + a_2 x^2 = -\frac{1}{30}x^5 + \frac{1}{2}x^4 - x^2 + \frac{5}{3}x - \frac{5}{4} + \int_0^x (x - s)(a_0 + a_1 s + a_2 s^2)ds + \int_0^x (x + s)(a_0 + a_1 s + a_2 s^2)ds
\]

Using some mathematical software, we can solve the unknown constants in Eq. (6), which read

\[
a_0 = -2, \quad a_1 = 0 \quad \text{and} \quad a_2 = 1
\]

An approximate solution is obtained as

\[
y(x) = x^2 - 2
\]

which happens to be the exact solution.

**Example 2** [8]. Consider the following Volterra-Fredholm integral equation

\[
y(x) = e^x - \frac{1}{3}e^{2x} + \frac{1}{3} + \int_0^x y(s)ds
\]

We assume that the solution can be approximated as

\[
y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3
\]

Putting Eq. (10) into Eq. (9) results in

\[
a_0 + a_1 x + a_2 x^2 + a_3 x^3 = e^x - \frac{1}{3}e^{2x} + \frac{1}{3} + \int_0^x (a_0 + a_1 s + a_2 s^2 + a_3 s^3)ds
\]

Expanding \( e^x \) and \( e^{2x} \)

\[
e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots
\]

\[
e^{2x} = 1 + 2x + \frac{9}{2}x^2 + \frac{27}{6}x^3 + \cdots
\]

and ignoring the terms \( x^n (n \geq 4) \), we re-write Eq. (11) in the form

\[
a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{3}(1 + 3x + \frac{9}{2}x^2 + \frac{27}{6}x^3) + \frac{1}{3} + a_1 x + \frac{3}{2}a_2 x^2 + a_3 x^3 + \frac{3}{2}a_2^2 x^3 + a_3 a_1 x^4 + \frac{3}{2}a_2 a_1 x^4 + \cdots
\]

From above equation, we identify that

\[
a_0 = 1
\]

\[
a_1 = a_0
\]

\[
a_2 = \frac{1}{2} + \frac{3}{2}a_1
\]

The constants can be determined as \( a_0 = 1, \quad a_1 = 1, \quad \text{and} \quad a_2 = 1/2 \), as a result an approximate solution is obtained, which is

\[
y(x) = 1 + x + \frac{1}{2}x^2
\]

The exact solution is \( y(x) = e^x \).

Considering \( y(0) = 1 \), we can assume the solution has the form

\[
y(x) = e^{ax}
\]
From Eq. (9), we have

\[ e^{\beta x} = e^\beta - \frac{1}{3} e^{3\beta x} + \frac{1}{3} \int_0^x e^{3\beta s} \, ds \tag{20} \]

or

\[ e^{\beta x} = e^\beta - \frac{1}{3} e^{3\beta x} + \frac{1}{3b}(e^{3\beta x} - 1) \tag{21} \]

Solving \( b \) from Eq. (21), we obtain

\[ b = 1 \tag{22} \]

We, therefore, obtain the following approximate solution

\[ y(x) = e^\beta \tag{23} \]

which is the exact one.

3. Conclusion

This paper suggests a simple analytical method for integral equation, a suitable choice of a trial solution always leads to an ideal result.

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